Mathematical Induction

Part Two

Problem Set Five

- Problem Set Four was due at 2:30PM today.
- Problem Set Five goes out today. It’s due next Friday at 2:30PM.
    - Play around with everything we’ve covered so far, plus a healthy dose of induction and inductive problem-solving.

Recap from Last Time

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

…and it stays true…

$\boldsymbol{P(0)}$ **is true**

and

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Theorem: The sum of the first n powers of two is $2^n - 1$.

***Proof*:** Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

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New Stuff!

Induction in Practice

- Often, a proof by induction will not explicitly state $P(n)$.
- Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what $P(n)$ is;
  - that $P(0)$ is true; and that
  - whenever $P(k)$ is true, $P(k+1)$ is true,

  the proof is usually valid. In this class, you could err on the side of safety by always defining it, but it's not required.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction.

For our base case, we'll prove the theorem is true when $n = 0$. The sum of the first zero powers of two is zero, and $2^0 - 1 = 0$, so the theorem is true in this case.

For the inductive step, assume the theorem holds when $n = k$ for some arbitrary $k \in \mathbb{N}$. Then we have

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

So the theorem is true when $n = k+1$, completing the induction. ■

A Fun Application:
The Limits of Data Compression

Bitstrings

- A ***bitstring*** is a finite sequence of 0s and 1s.
- Examples:
  - 11011100
  - 010101010101
  - 0000
  - ε (the ***empty string***)
- There are 2^n bitstrings of length n.

Data Compression

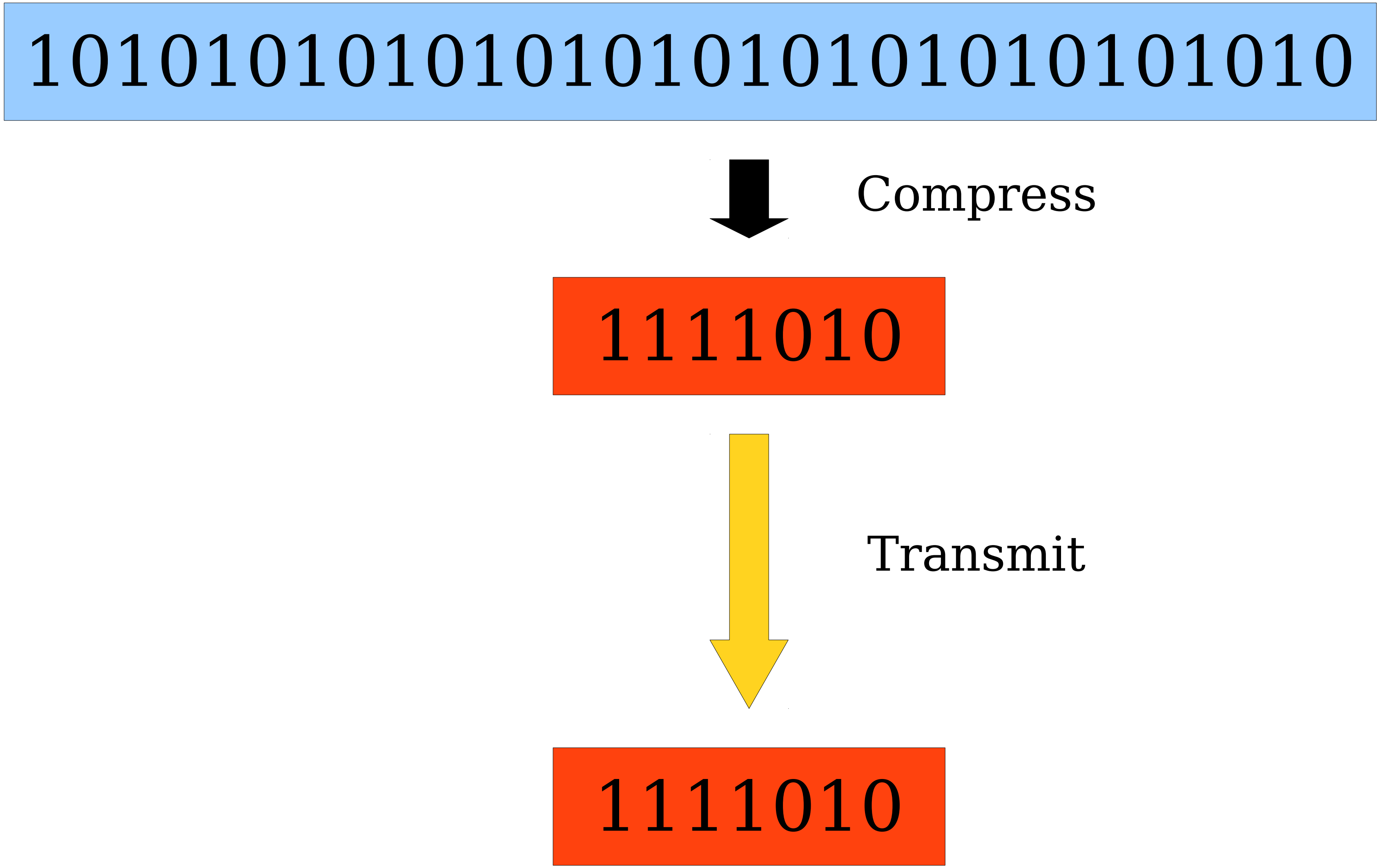
- Inside a computer, all data are represented as sequences of 0s and 1s (bitstrings)
- To transfer data over a network (or on a flash drive, if you're still into that), it is useful to reduce the number of 0s and 1s before transferring it.
- Most real-world data can be compressed by exploiting redundancies.
  - Text repeats common patterns ("the", "and", etc.)
  - Bitmap images use similar colors throughout the image.
- ***Idea:*** Replace each bitstring with a *shorter* bitstring that contains all the original information.
  - This is called ***lossless data compression***.

101010101010101010101010101010

101010101010101010101010101010

Compress

1111010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Transmit
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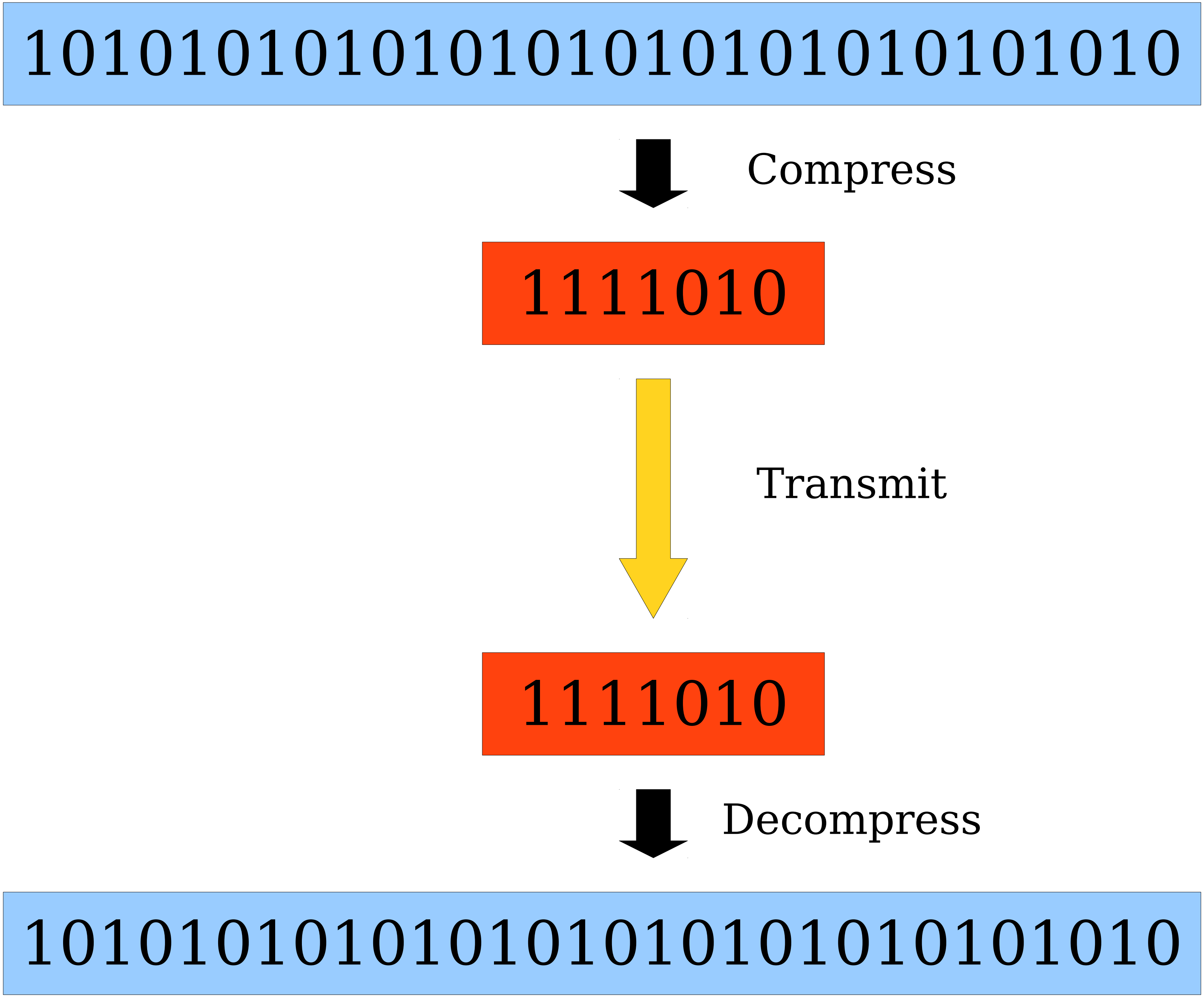
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Lossless Data Compression

- In order to losslessly compress data, we need two functions:
  - A ***compression function*** C, and
  - A ***decompression function*** D.
- We need to have $D(C(x)) = x$.
  - Otherwise, we can't uniquely encode or decode some bitstring.
- This means that D must be a left inverse of C, so (as you proved in PS3!) C must be injective.

A Perfect Compression Function

- Ideally, the compressed version of a bitstring would always be shorter than the original bitstring.
- ***Question***: Can we find a lossless compression algorithm that always compresses a string into a shorter string?
- To handle the issue of the empty string (which can't get any shorter), let's assume we only care about strings of length at least 10.

A Counting Argument

- Let $\mathbb{B}^n$ be the set of bitstrings of length n, and $\mathbb{B}^{<n}$ be the set of bitstrings of length less than n.
- How many bitstrings of length n are there?
  - ***Answer***: 2^n
- How many bitstrings of length *less than n* are there?
  - ***Answer***: $2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1$
- By the pigeonhole principle, no function from $\mathbb{B}^n$ to $\mathbb{B}^{<n}$ can be injective – at least two elements must collide!
- Since a perfect compression function would have to be an injection from $\mathbb{B}^n$ to $\mathbb{B}^{<n}$, ***there is no perfect compression function!***

Why this Result is Interesting

- Our result says that no matter how hard we try, it is ***impossible*** to compress every string into a shorter string.
- No matter how clever you are, you cannot write a lossless compression algorithm that always makes strings shorter.
- In practice, only highly redundant data can be compressed.
- The fields of ***information theory*** and ***Kolmogorov complexity*** explore the limits of compression; if you're interested, go explore!

Variations on Induction: ***Starting Later***

Induction Starting at 0

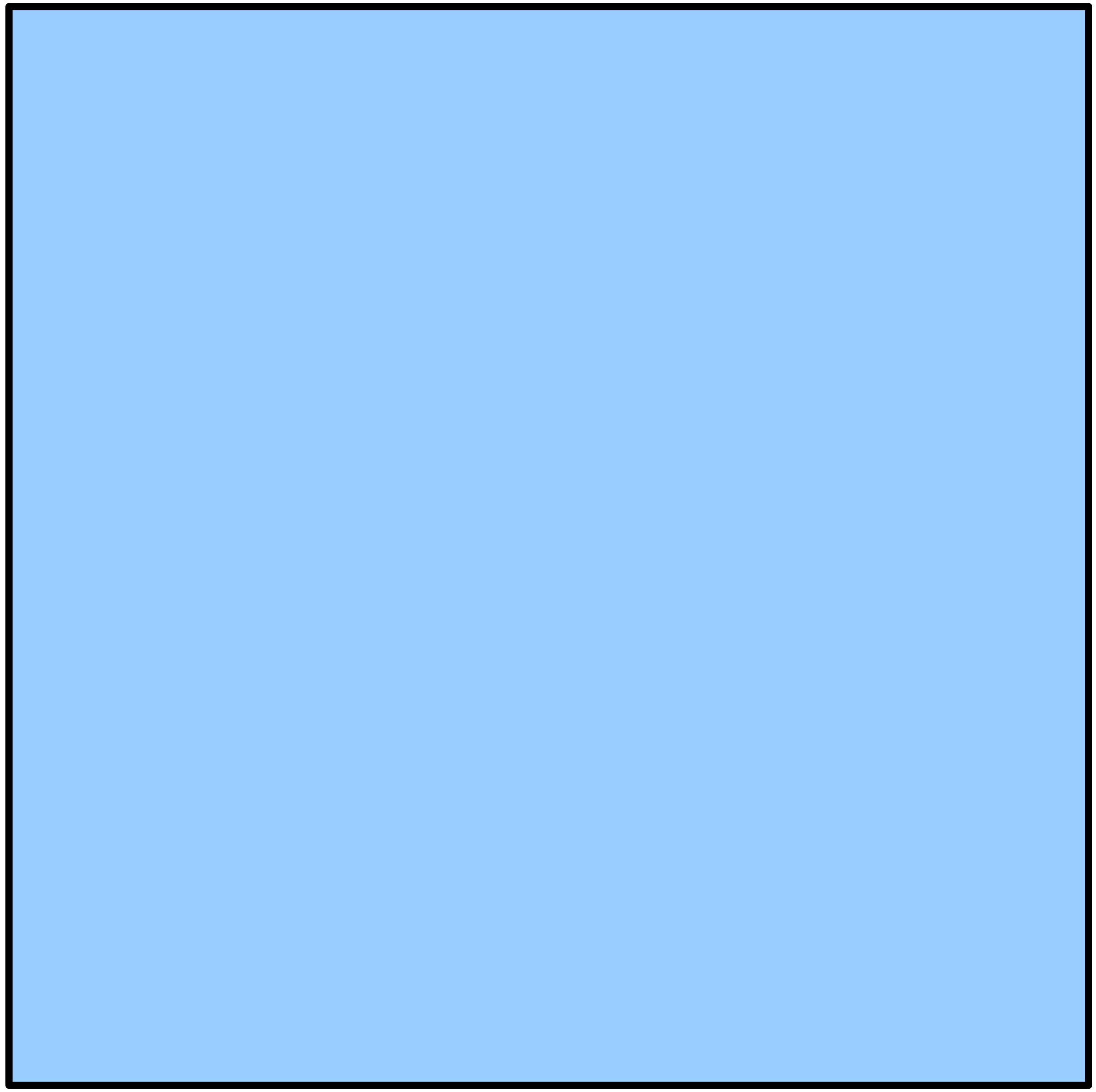
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $k \geq 0$, that
    if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

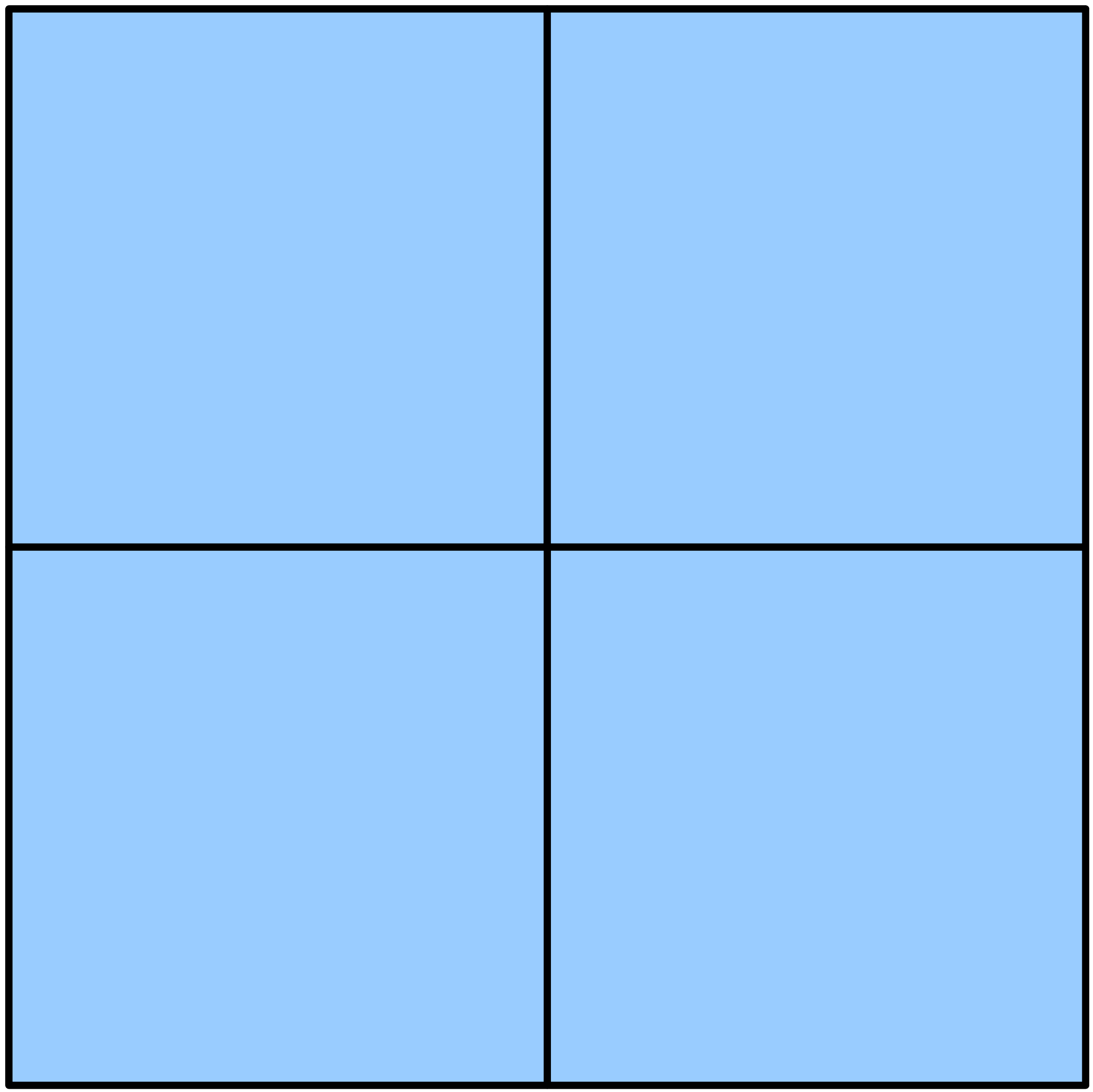
- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
  - Show that $P(\boldsymbol{m})$ is true.
  - Show that for any $k \geq \boldsymbol{m}$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

Variations on Induction: ***Bigger Steps***

Subdividing a Square



Subdividing a Square



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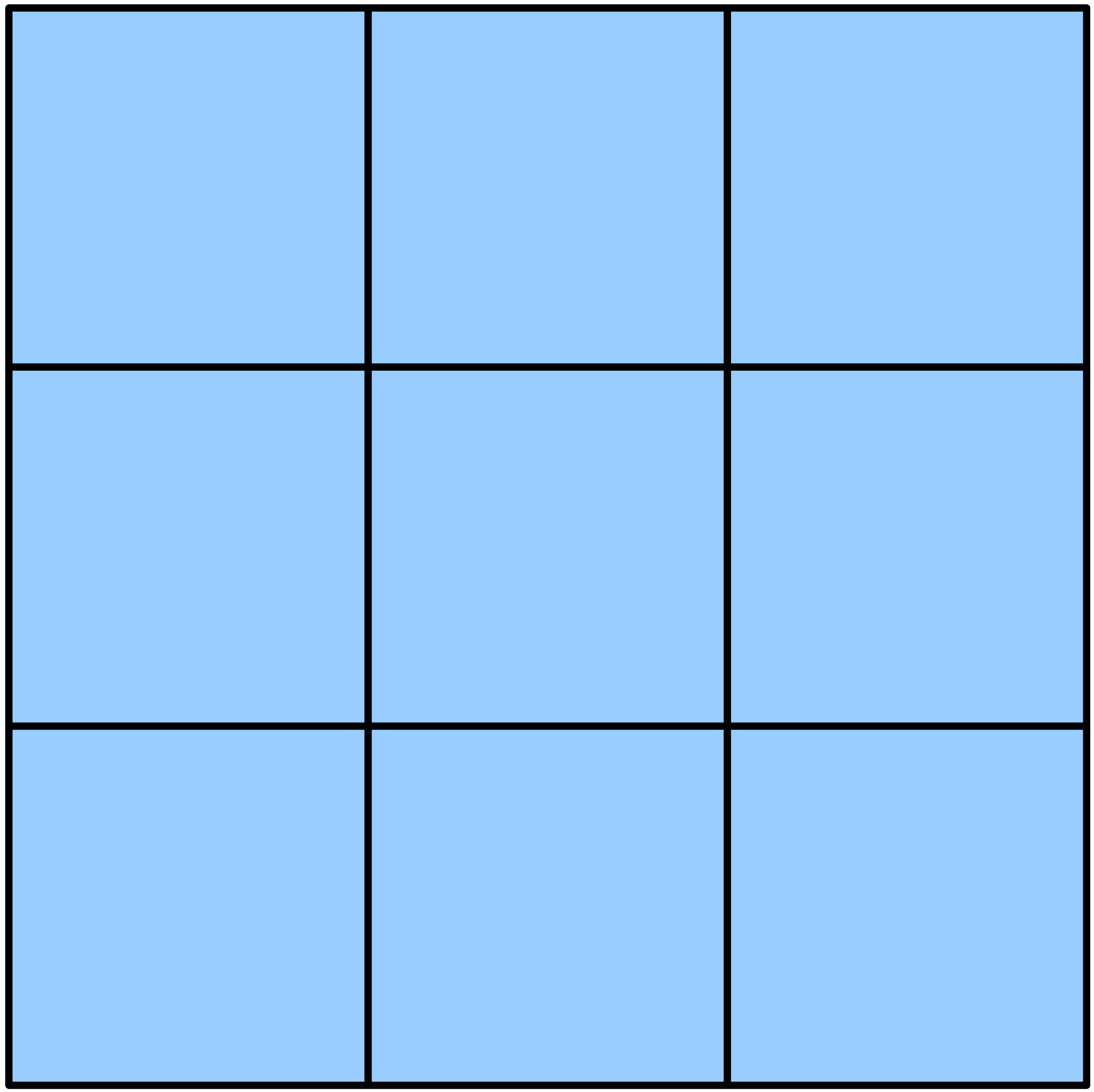

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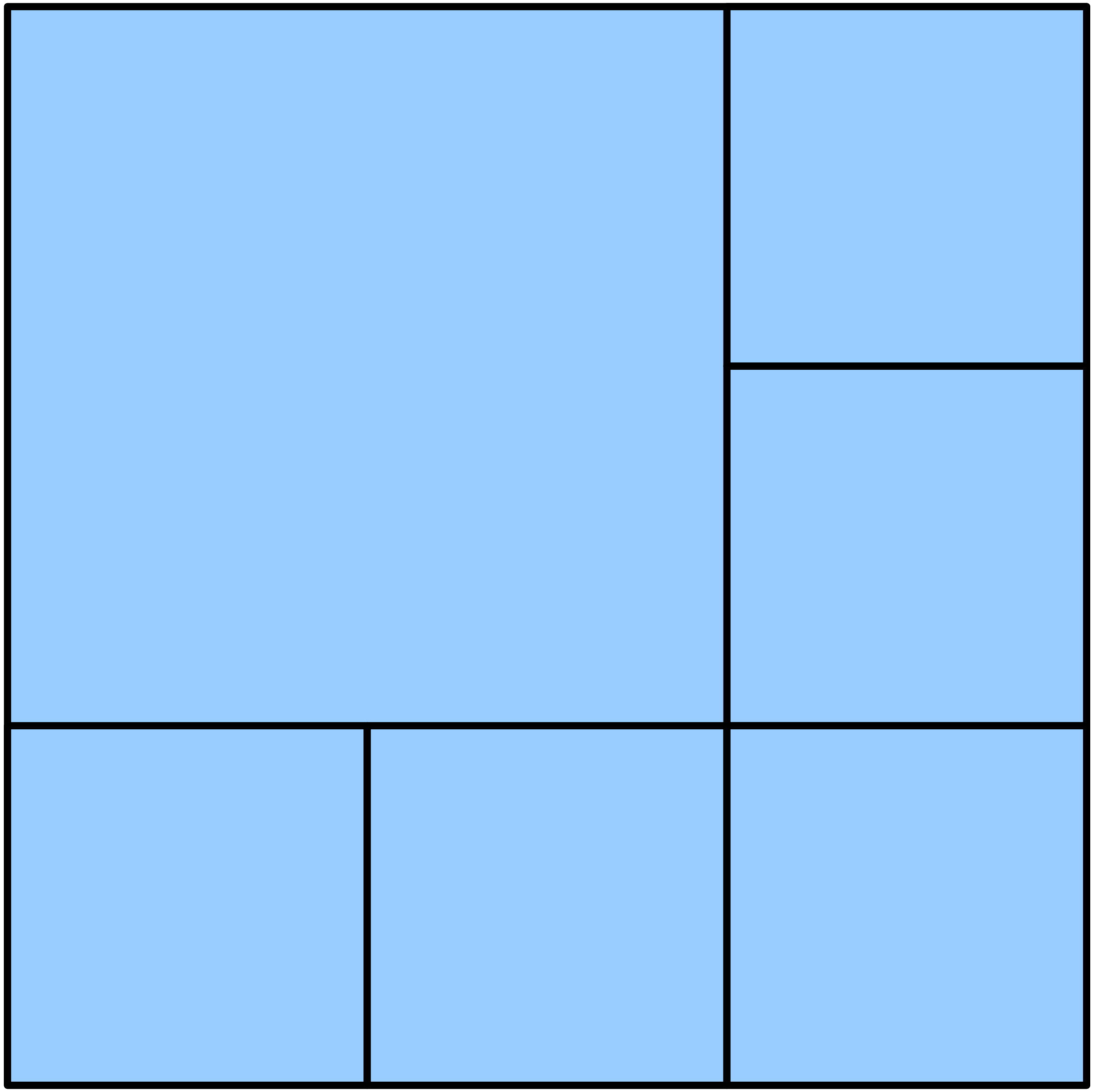

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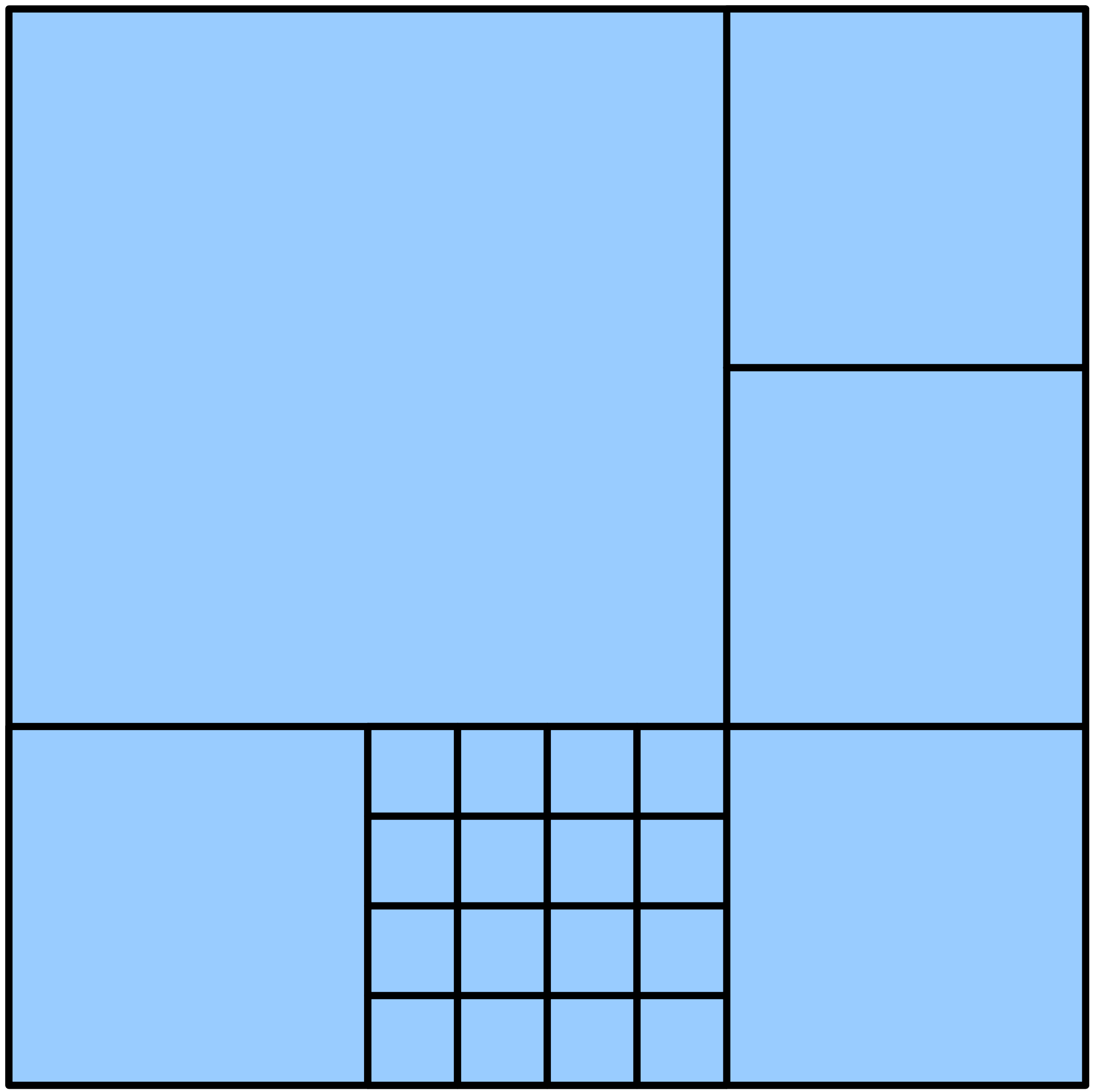

For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

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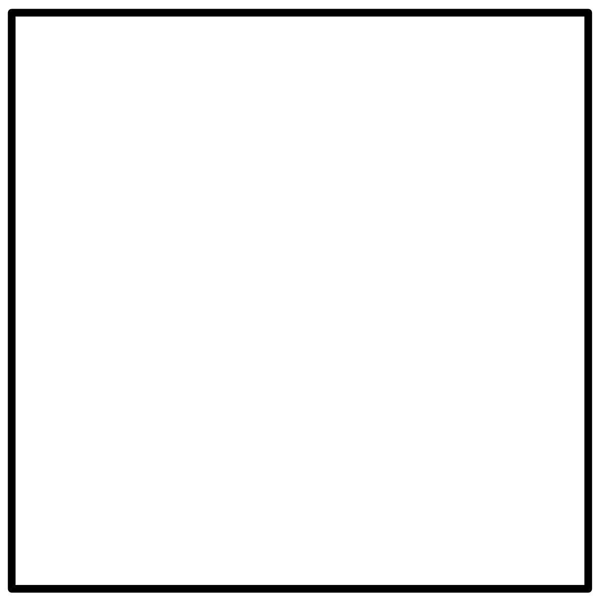
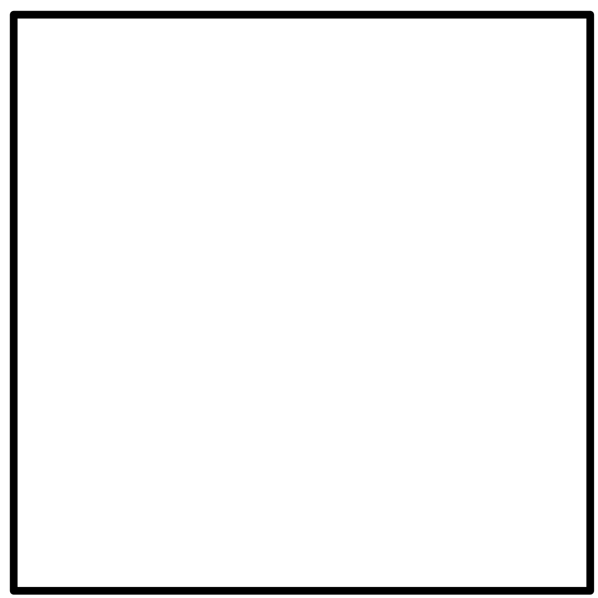
1

1 2 3 4 5 6 7 8 9 10 11 12



1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



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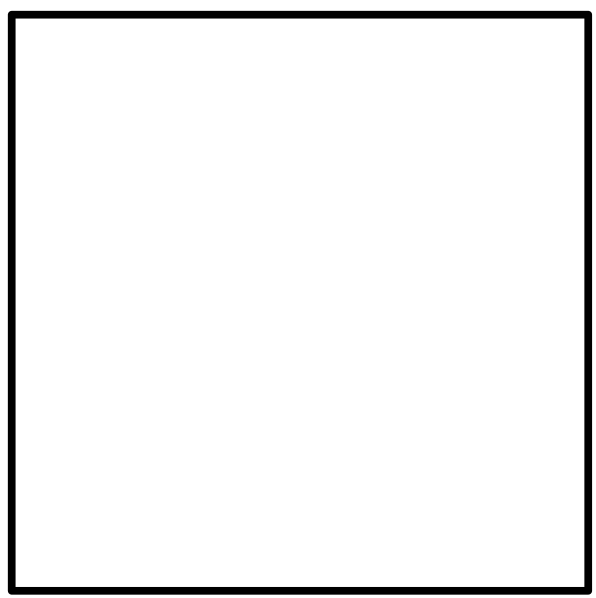
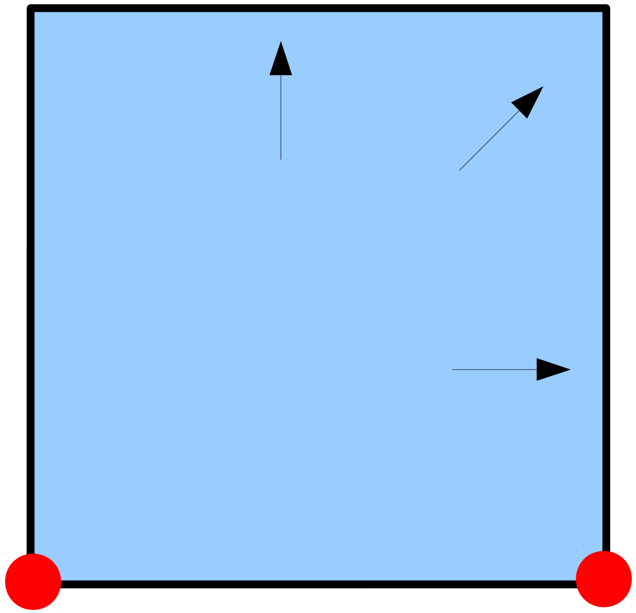


Number of corners = 4

Number of squares < 4

1 2 3 4 5 6 7 8 9 10 11 12

Each of the original corners needs to be covered by a corner of the new smaller squares.



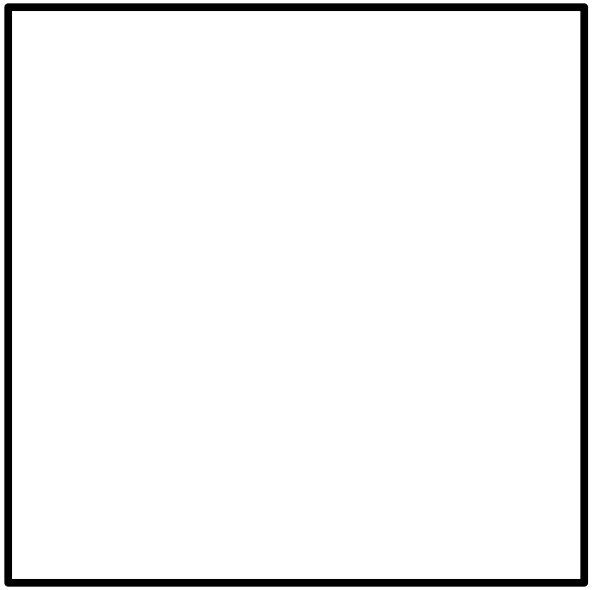
By the pigeonhole principle, at least one smaller square needs to cover at least *two* of the original square's corners.

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

1	2
4	3

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



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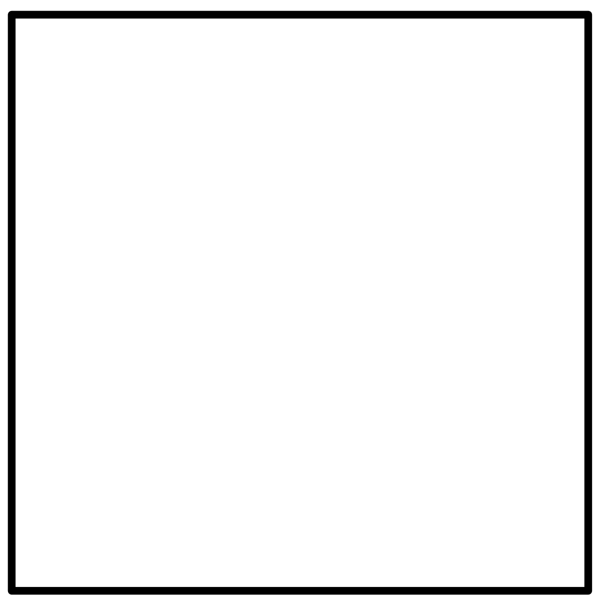
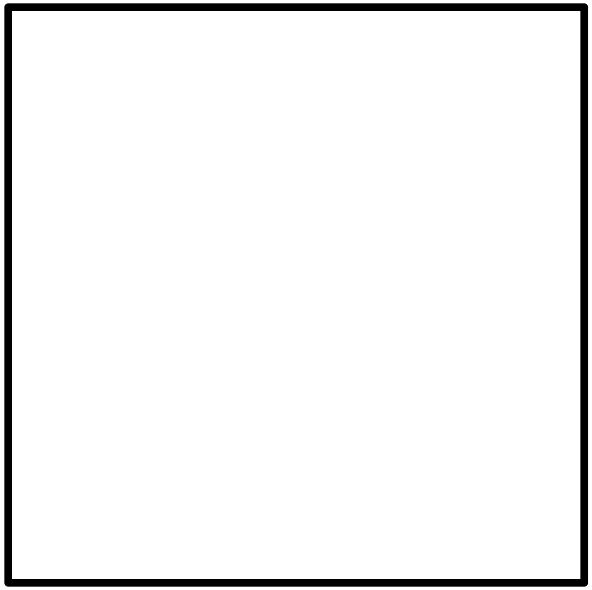


Number of corners = 4

Number of squares = 5

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12

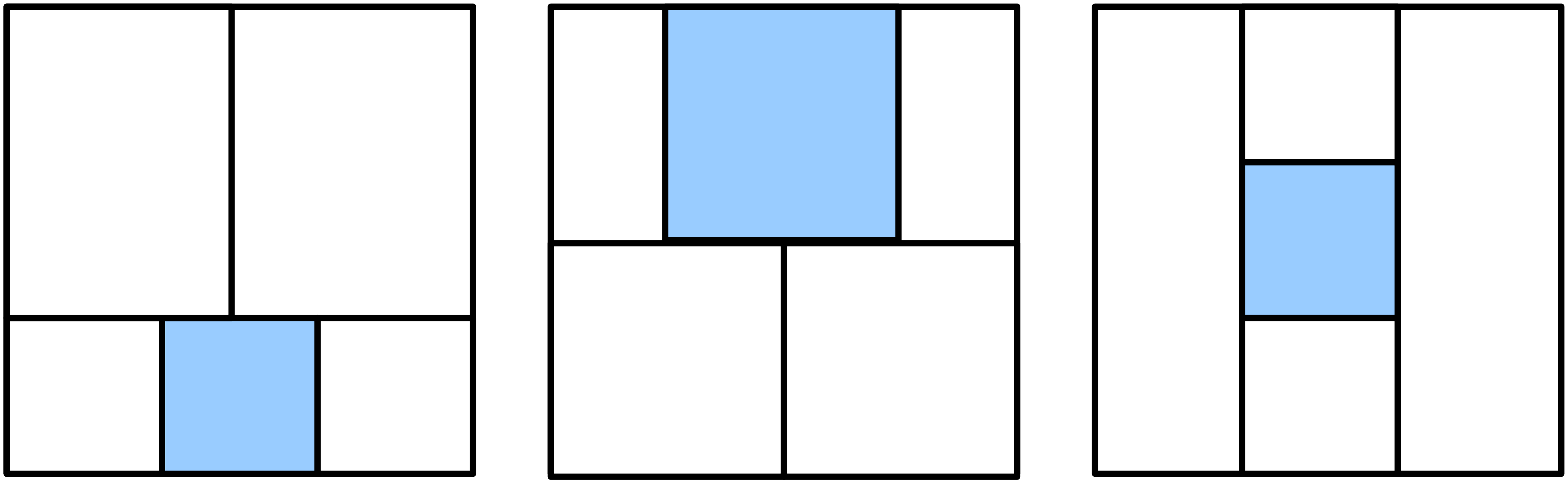
At least one square cannot be covering *any* of the original corners



Number of corners = 4

Number of squares = 5

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



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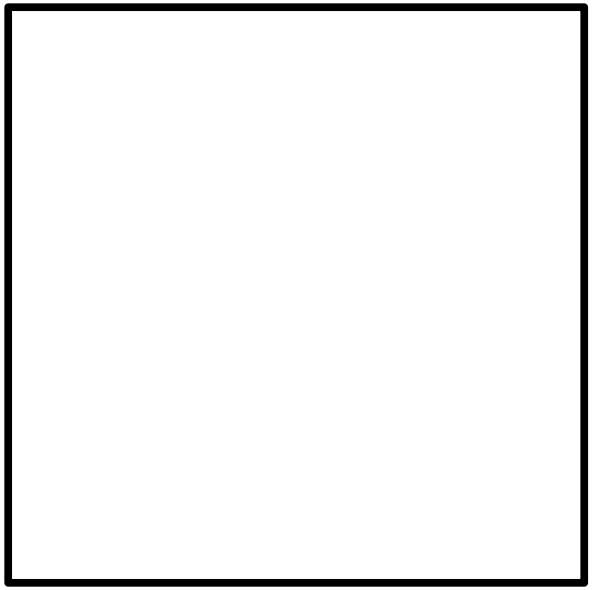

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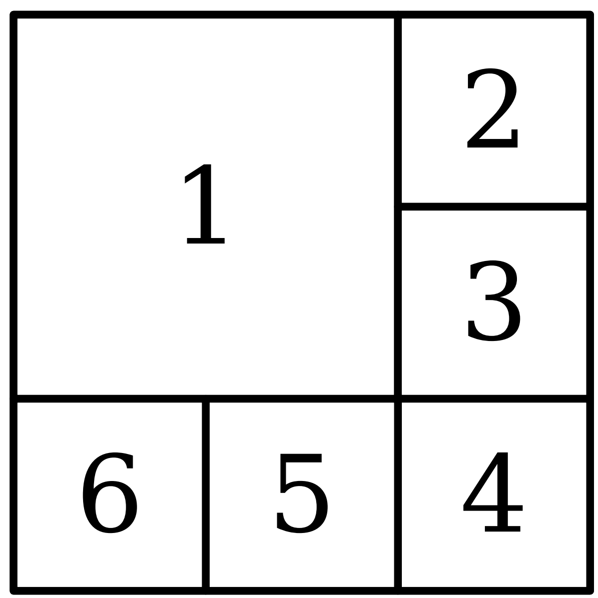

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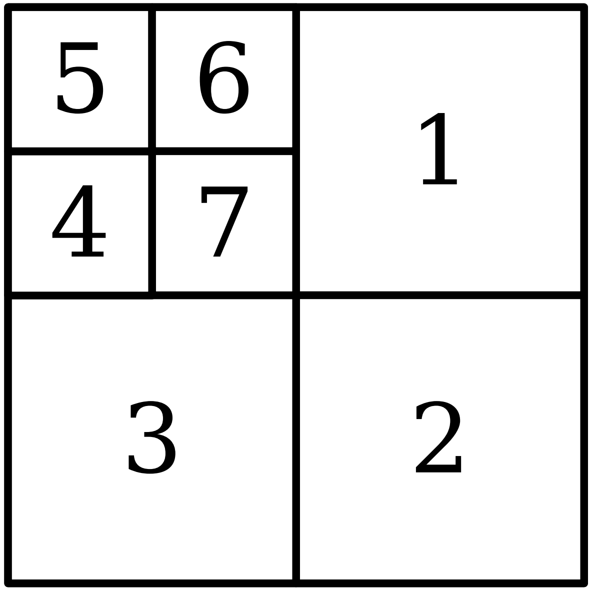

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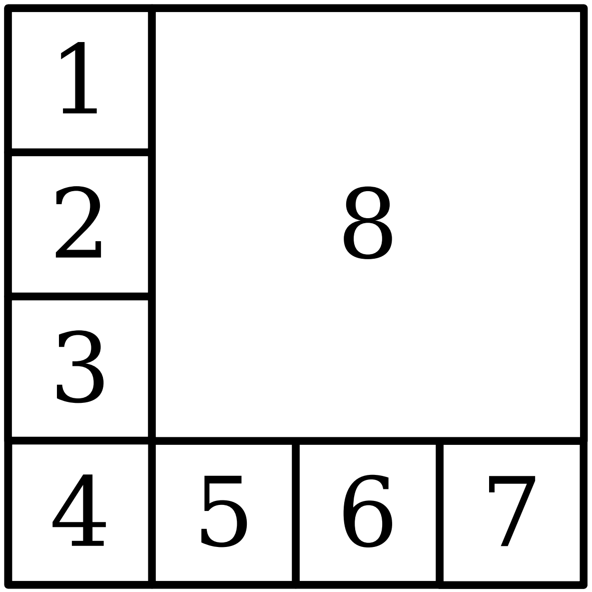

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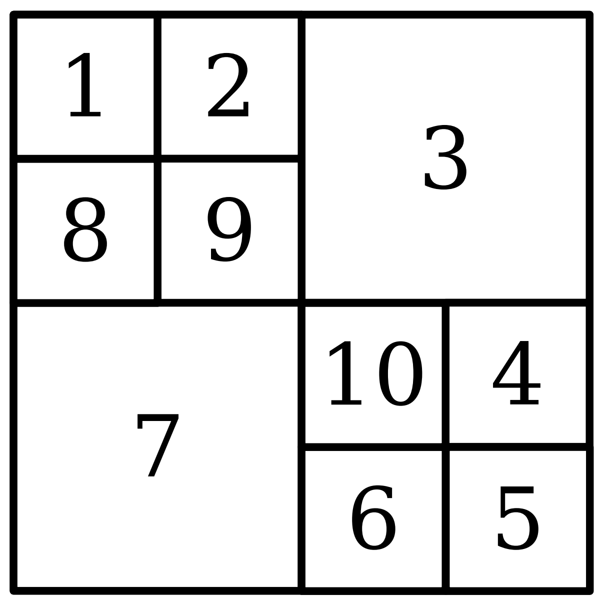

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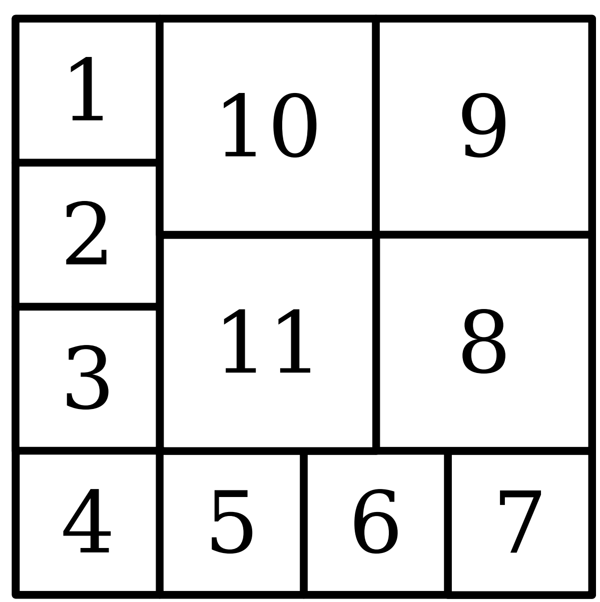

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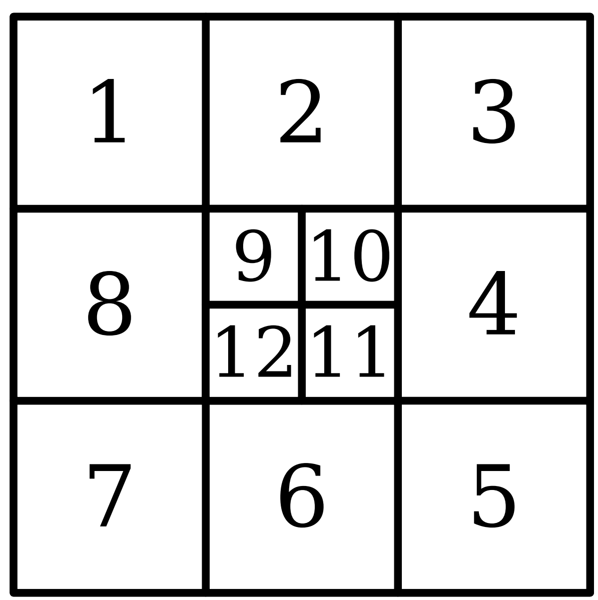

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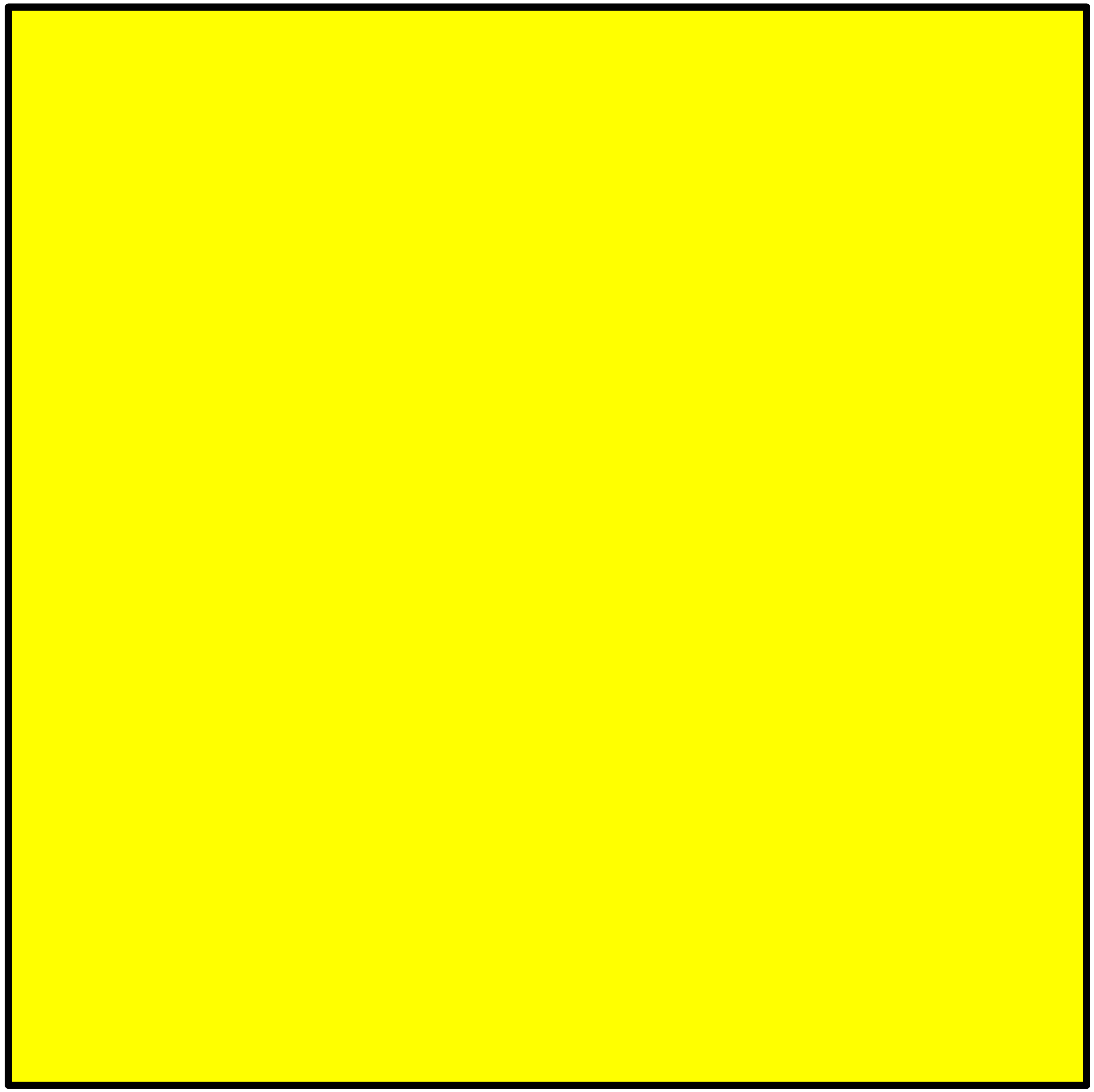

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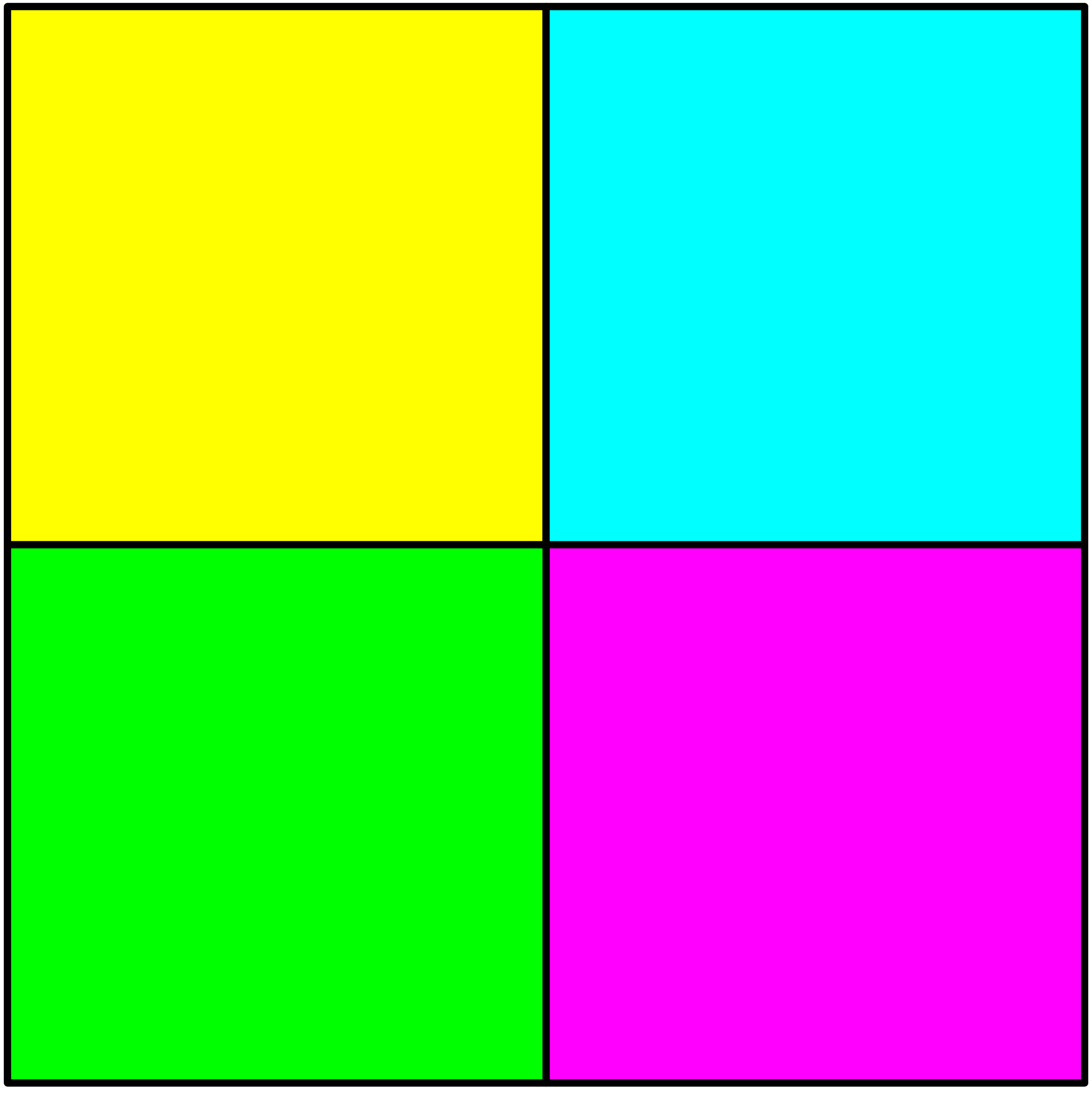

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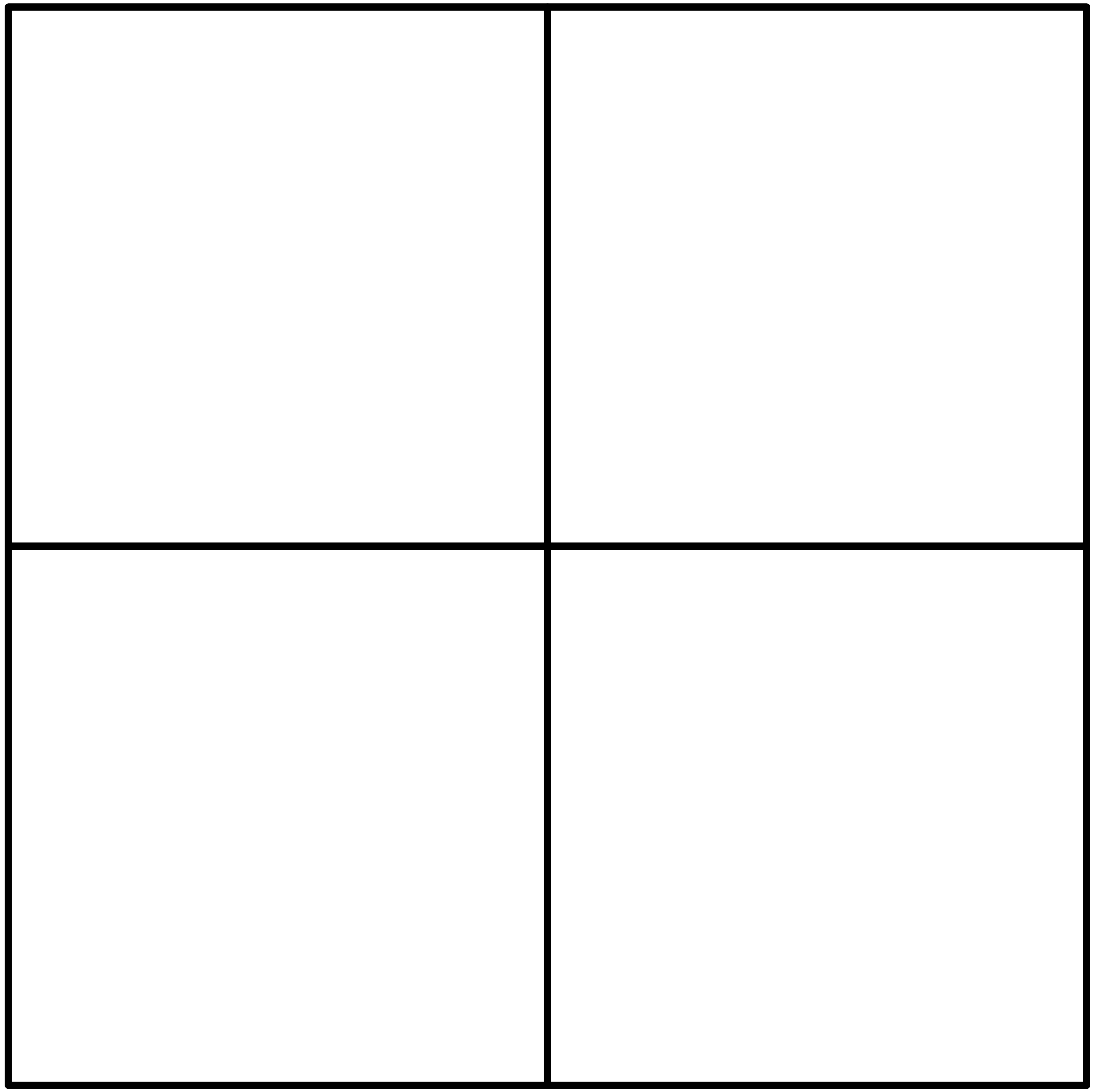

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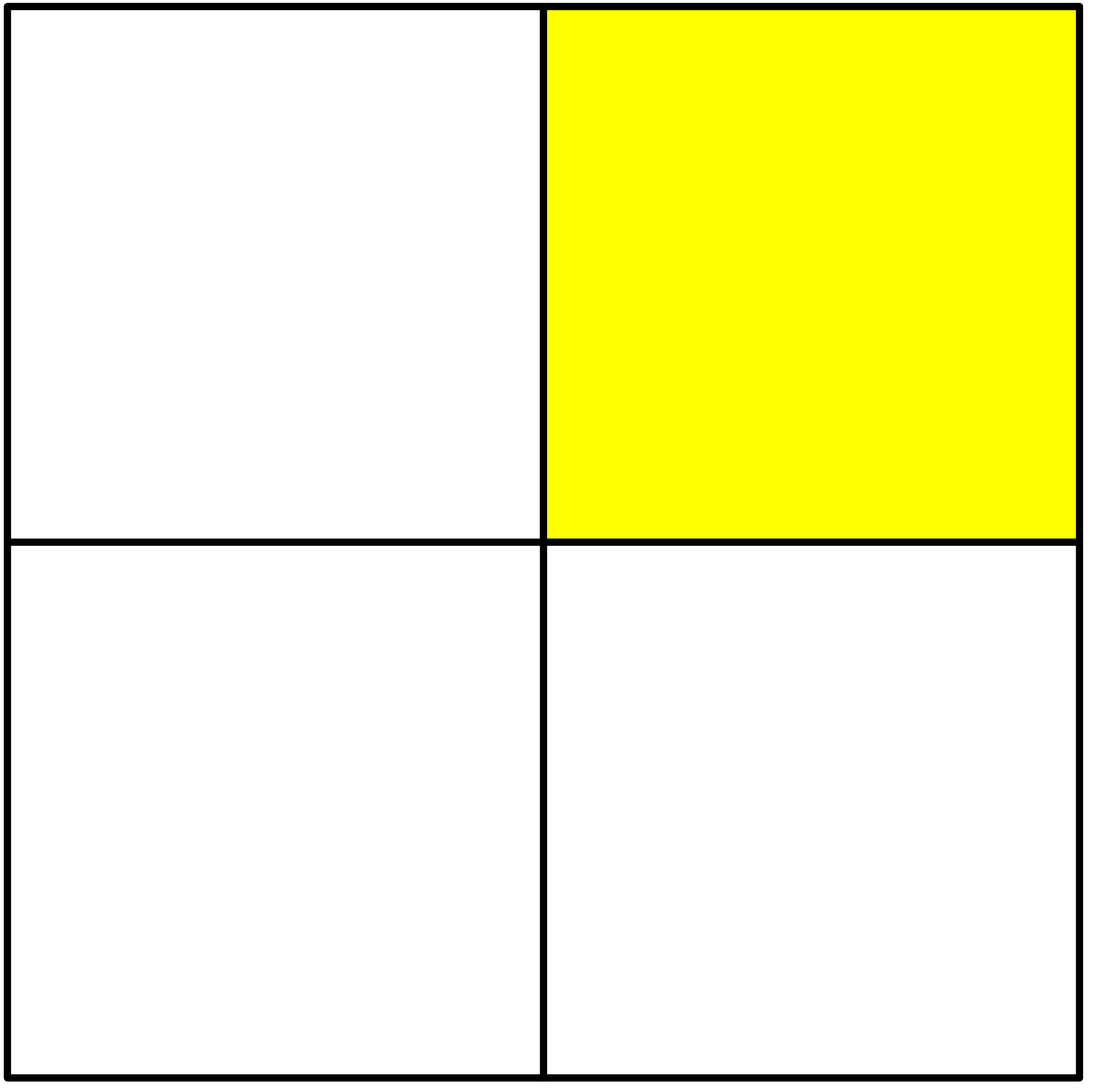

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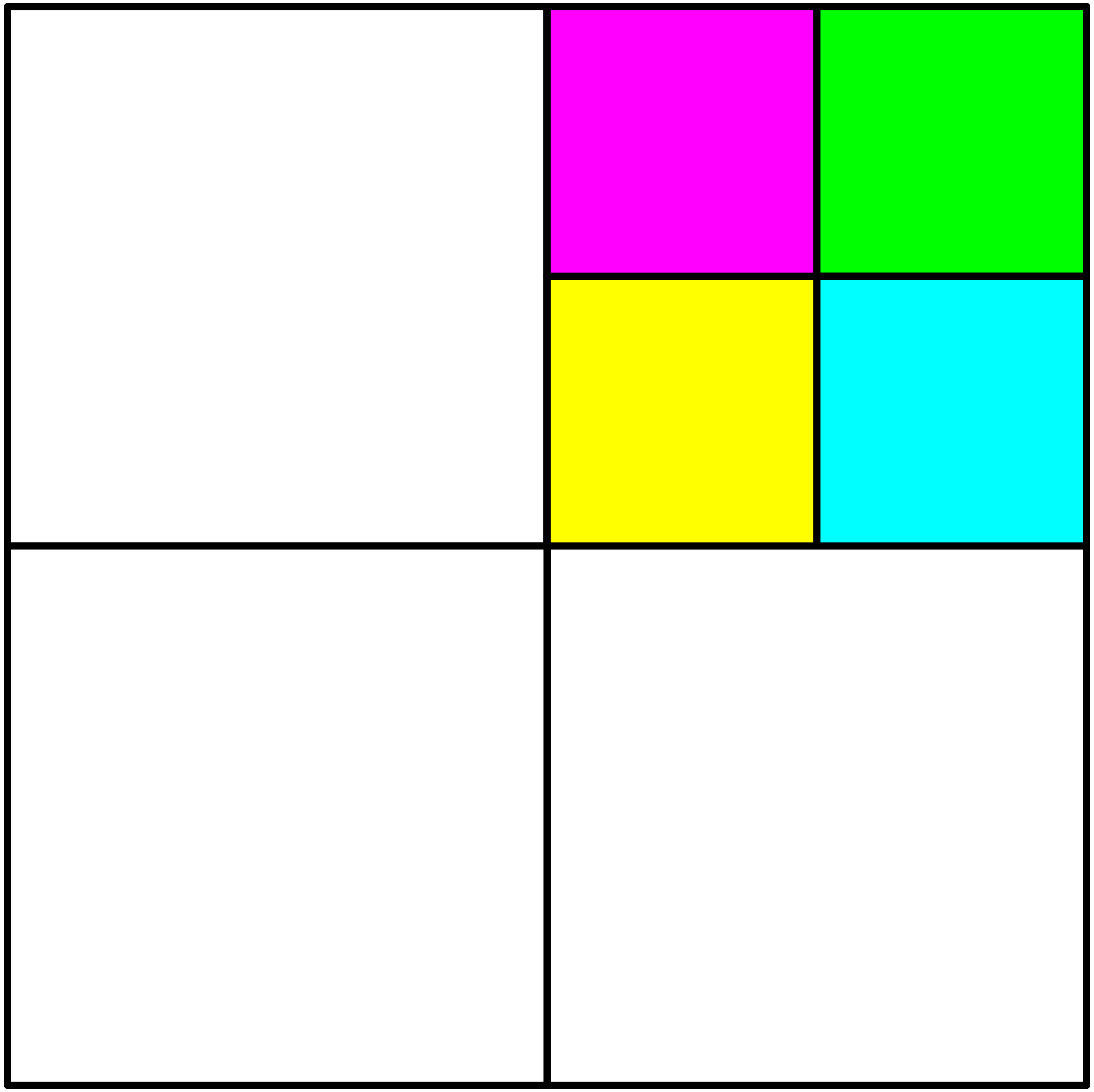

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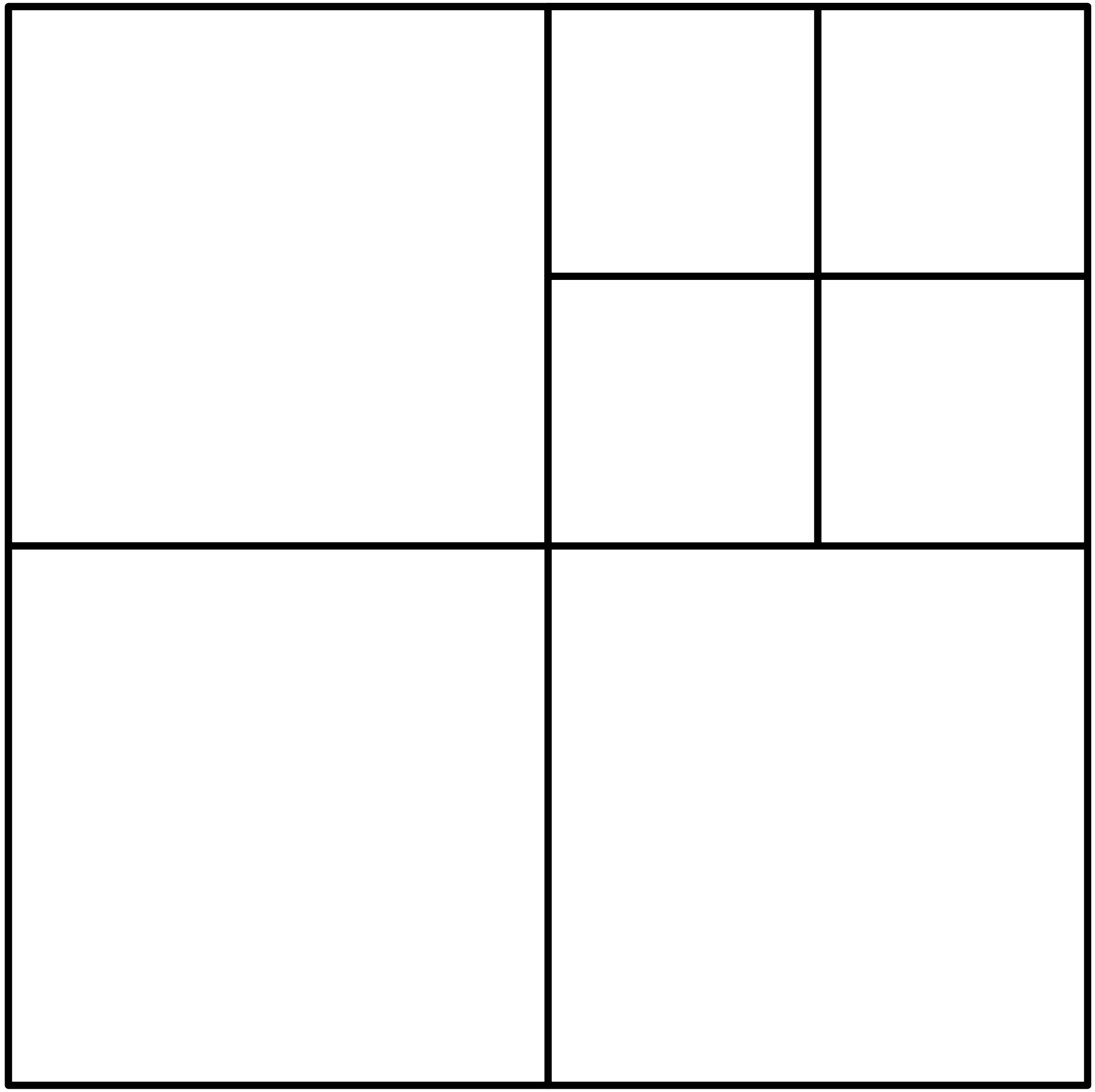

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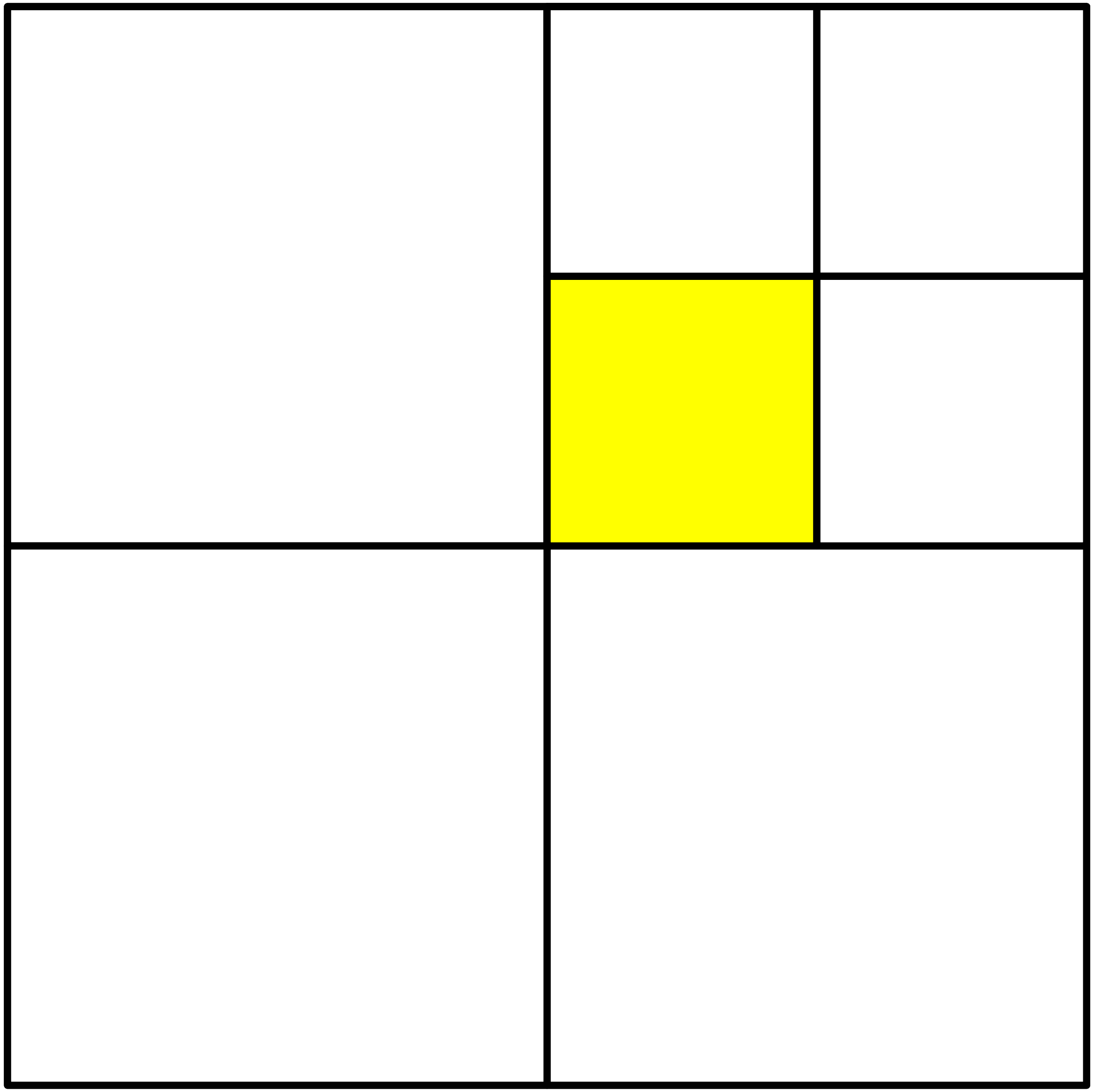

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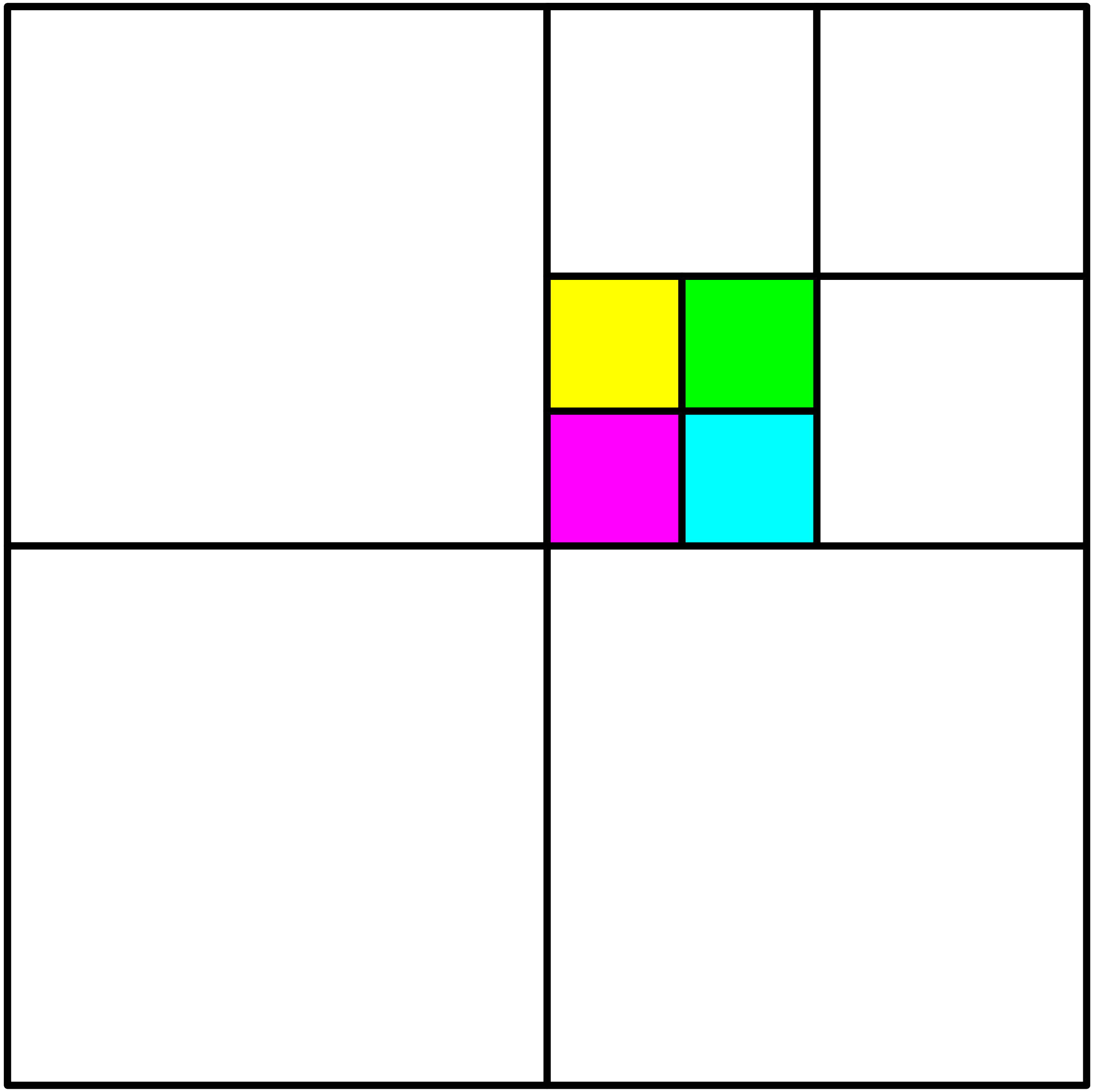

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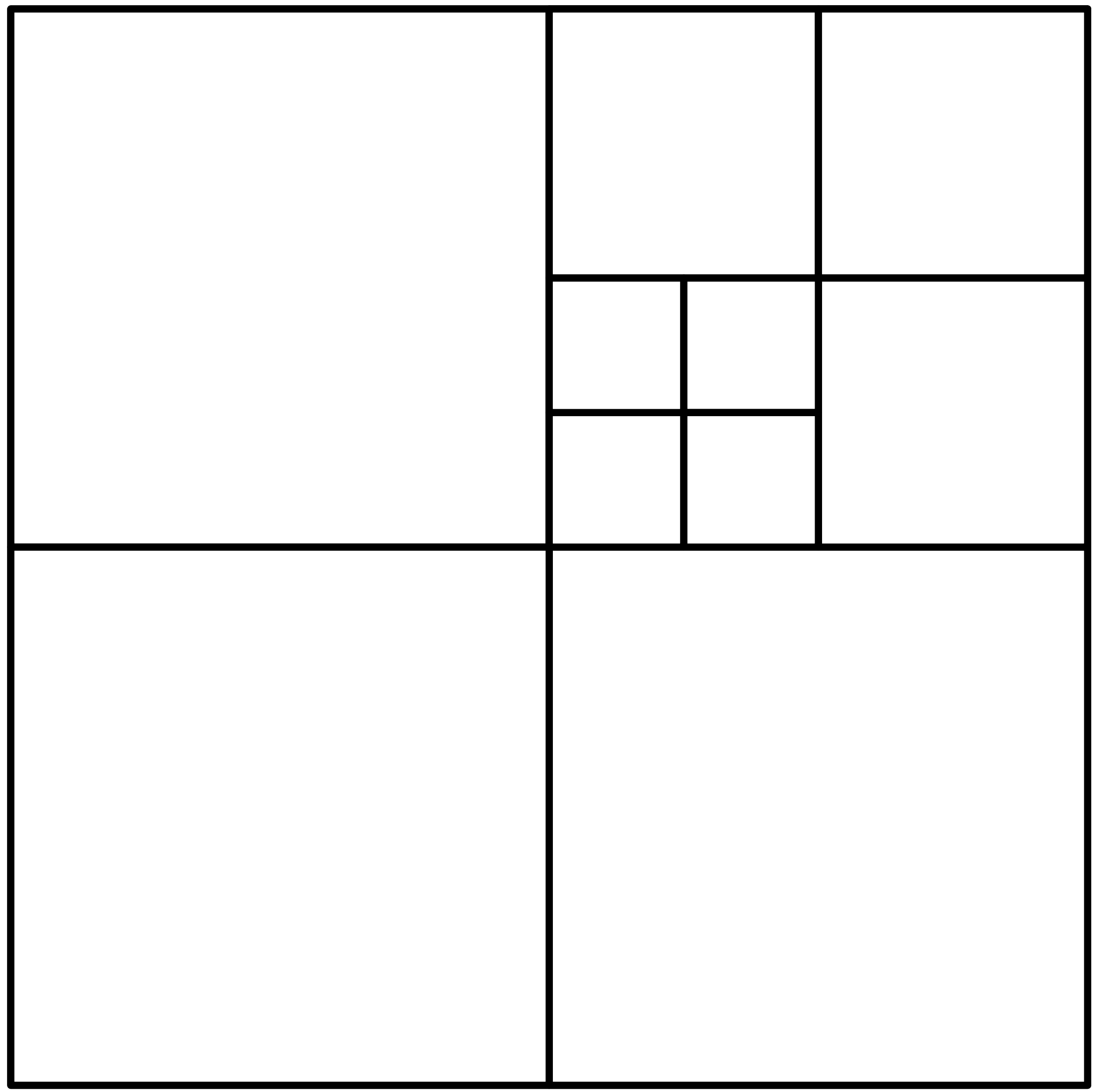

The Key Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
  - For multiples of three, start with 6 and keep adding three squares until n is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

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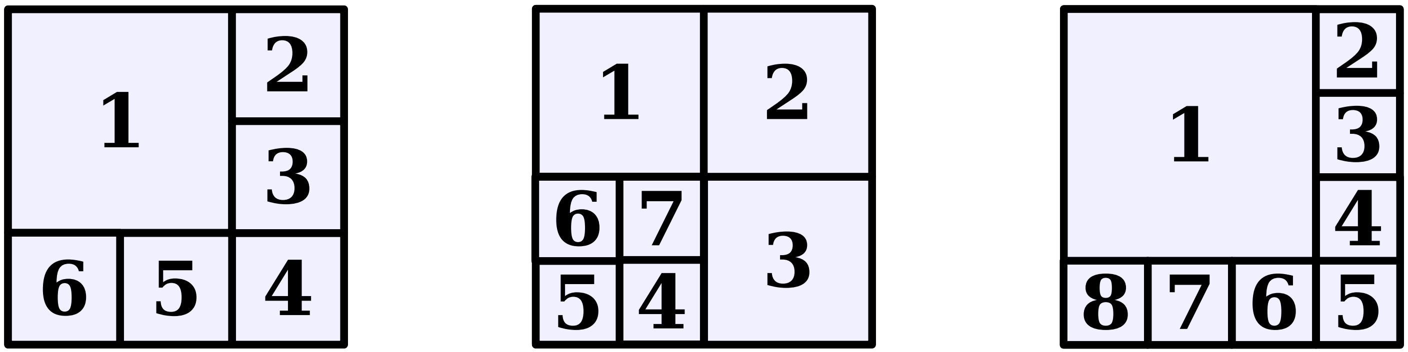

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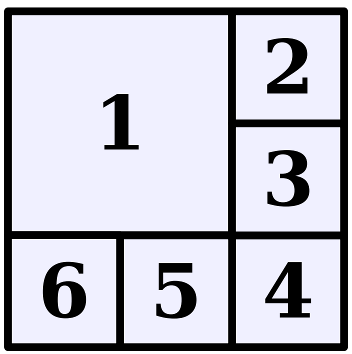

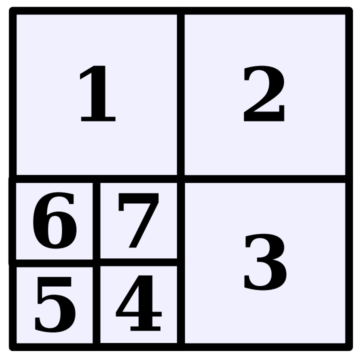

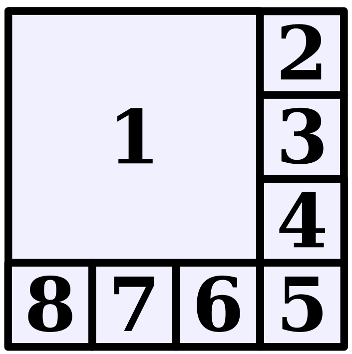

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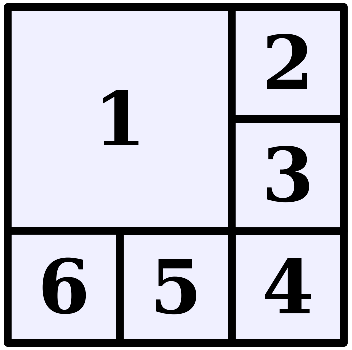

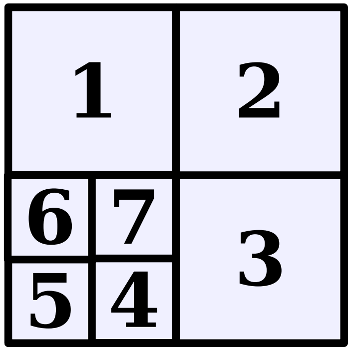

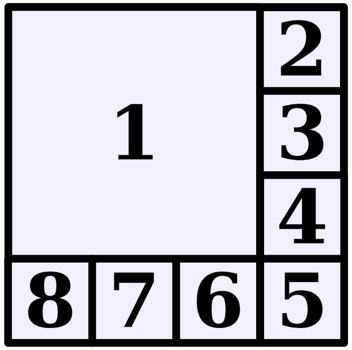

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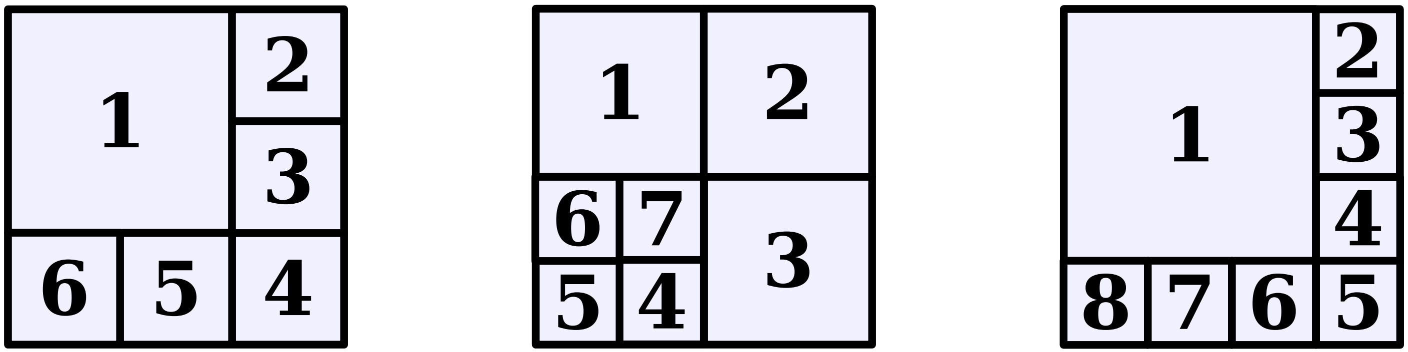

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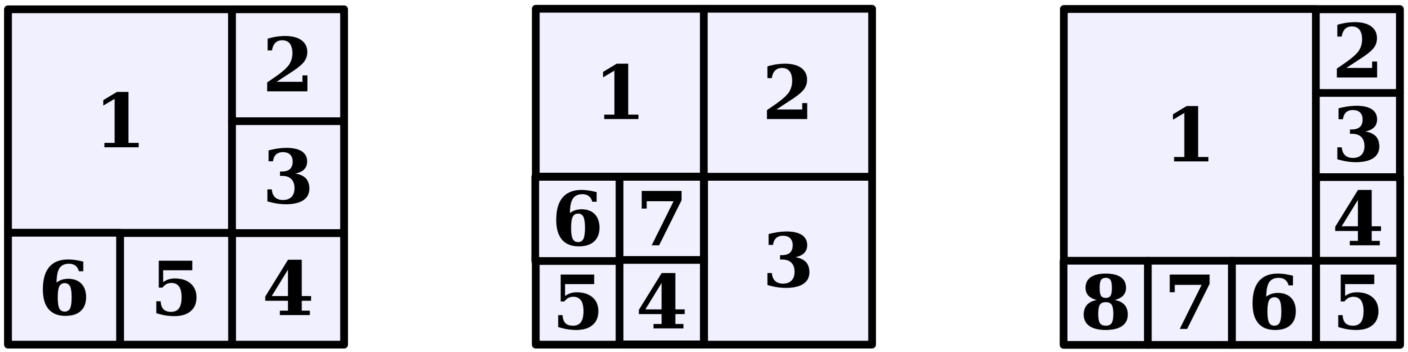

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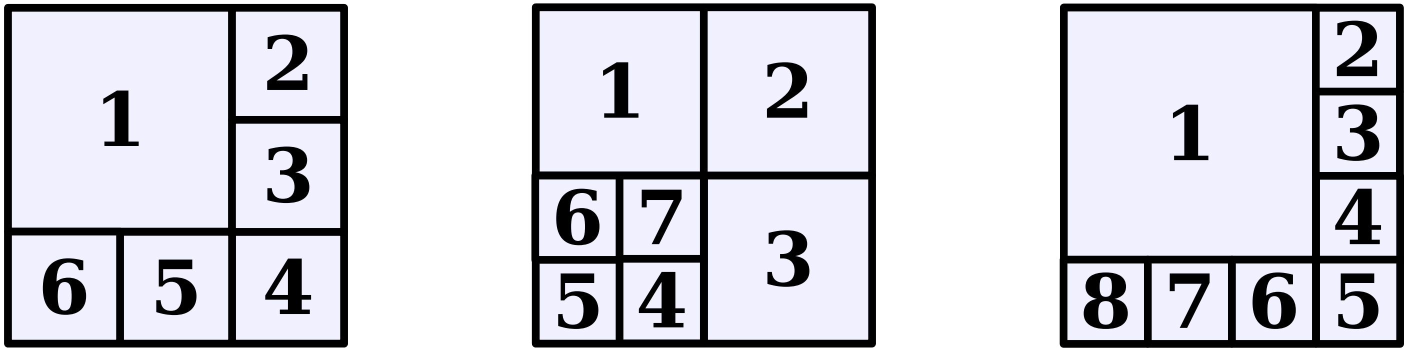

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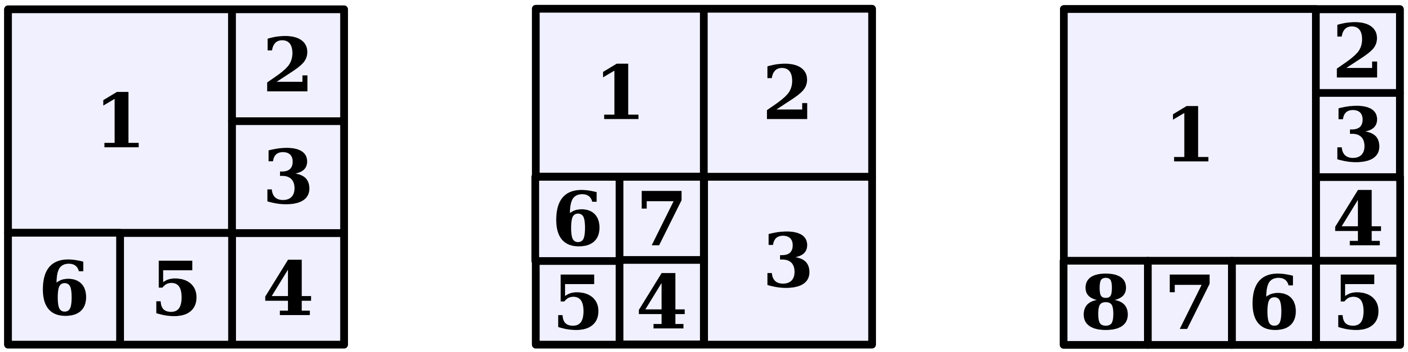

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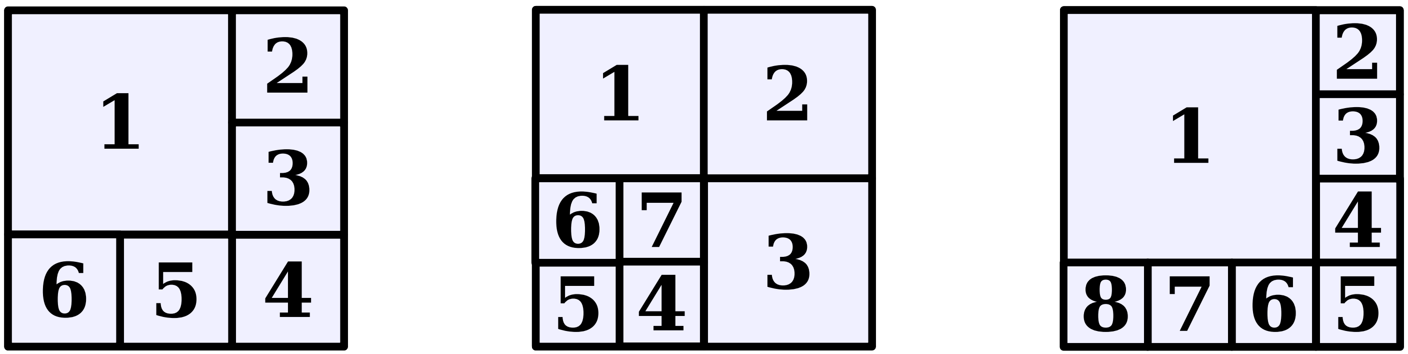

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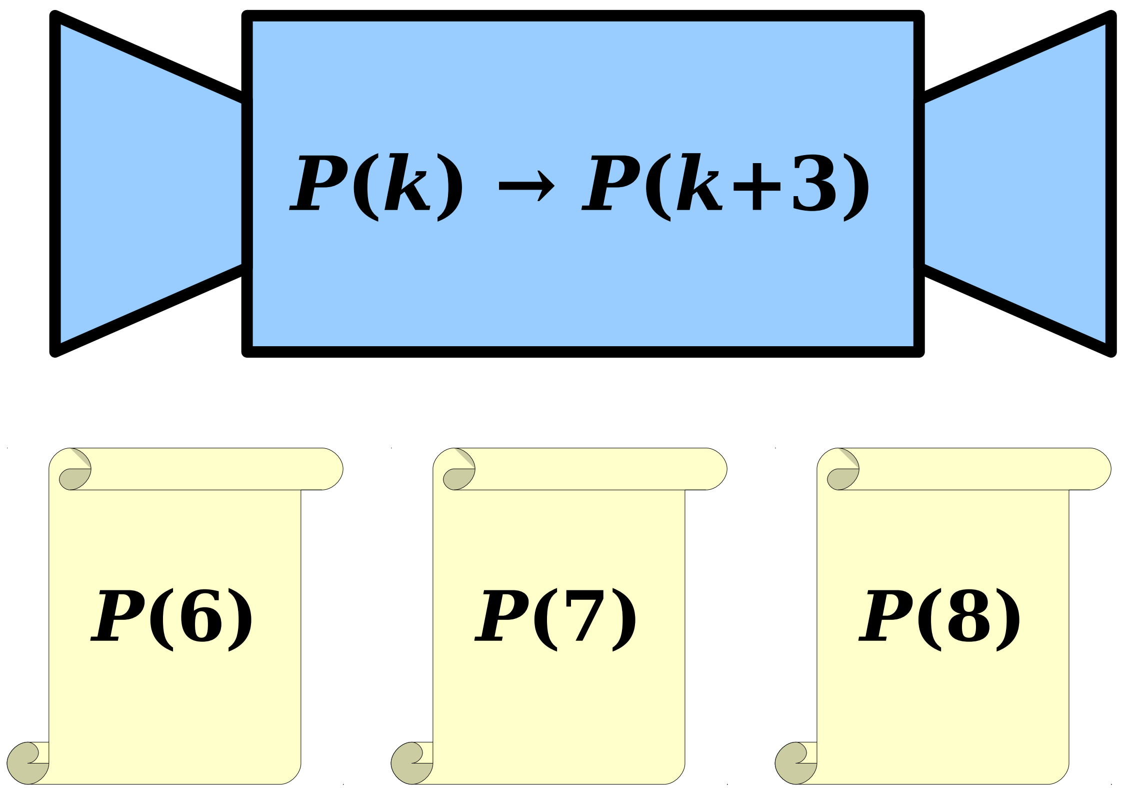
- Thinking back to our “induction machine” analogy:



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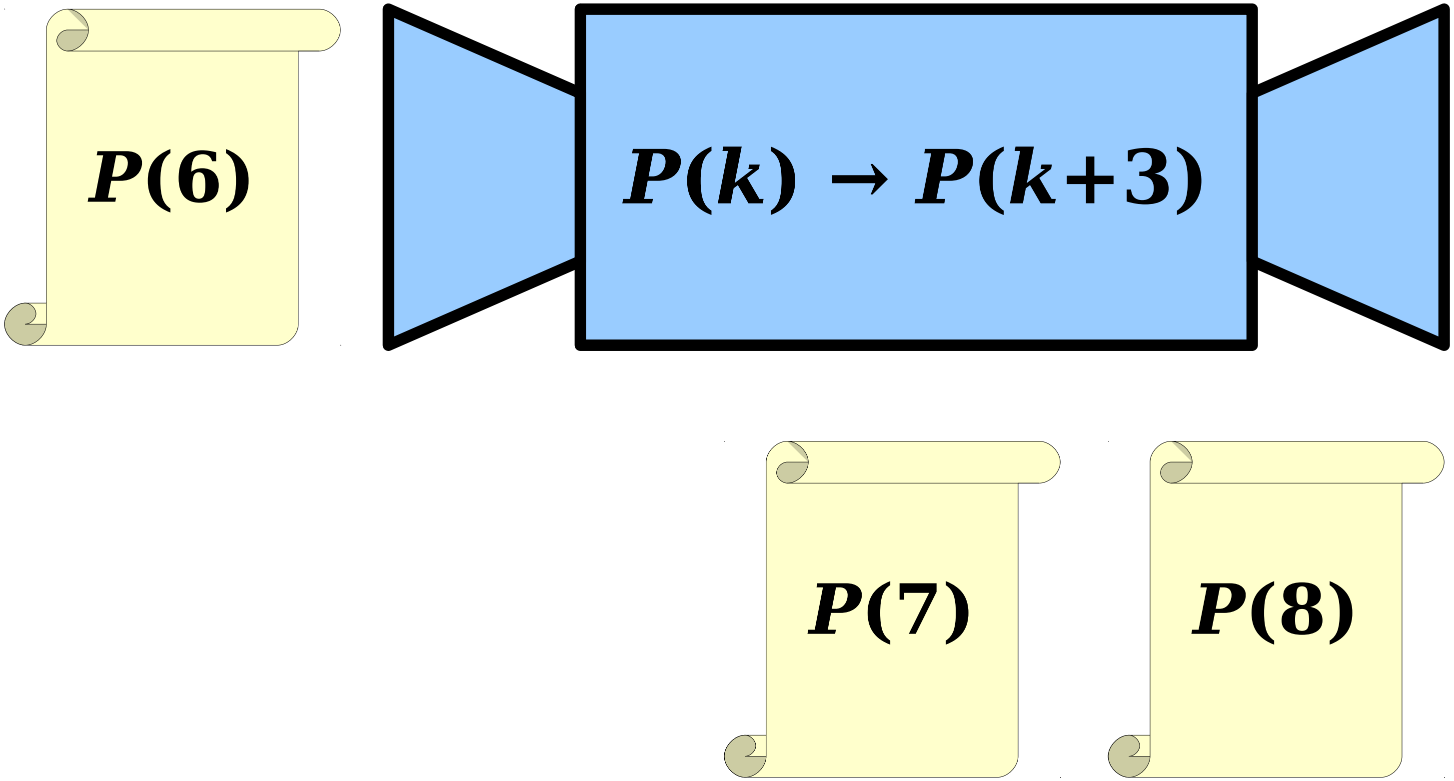

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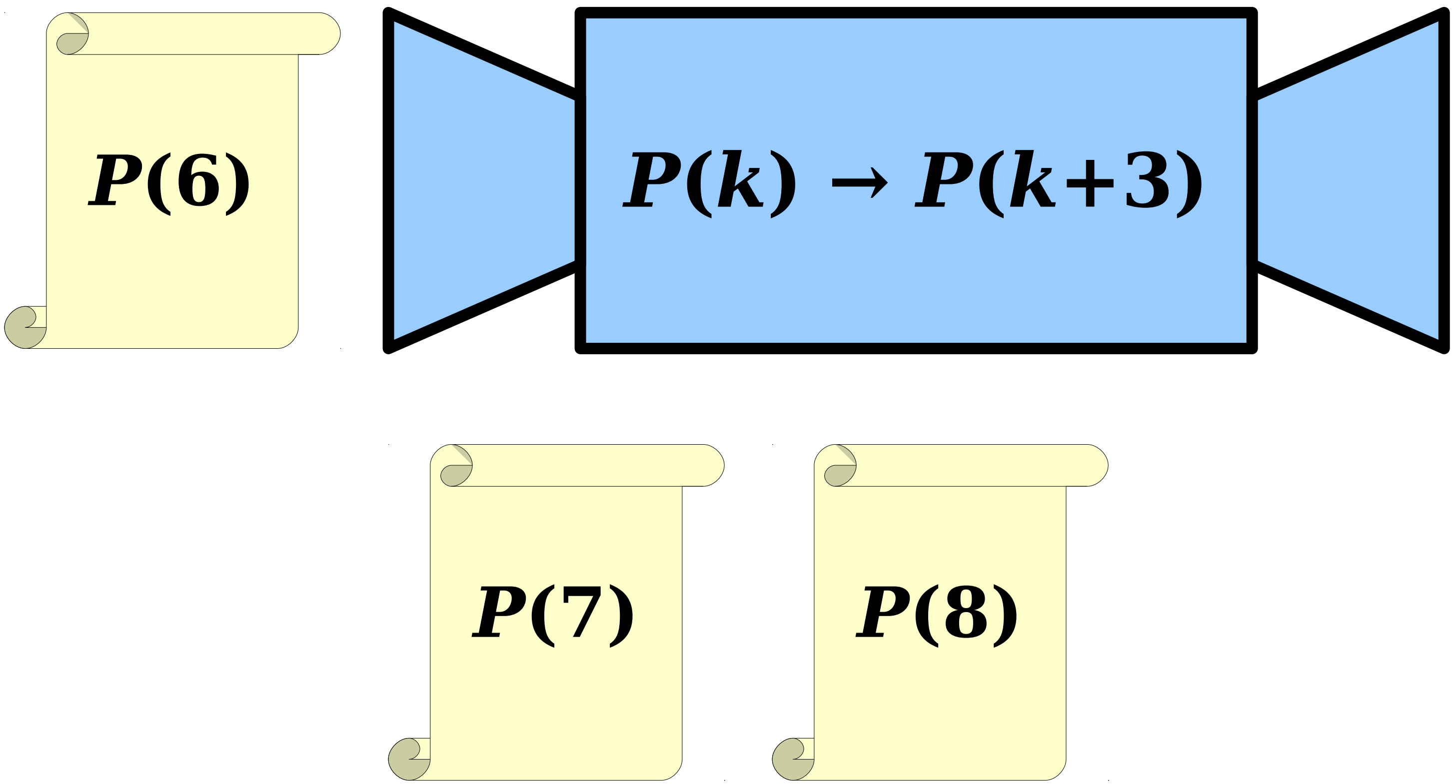

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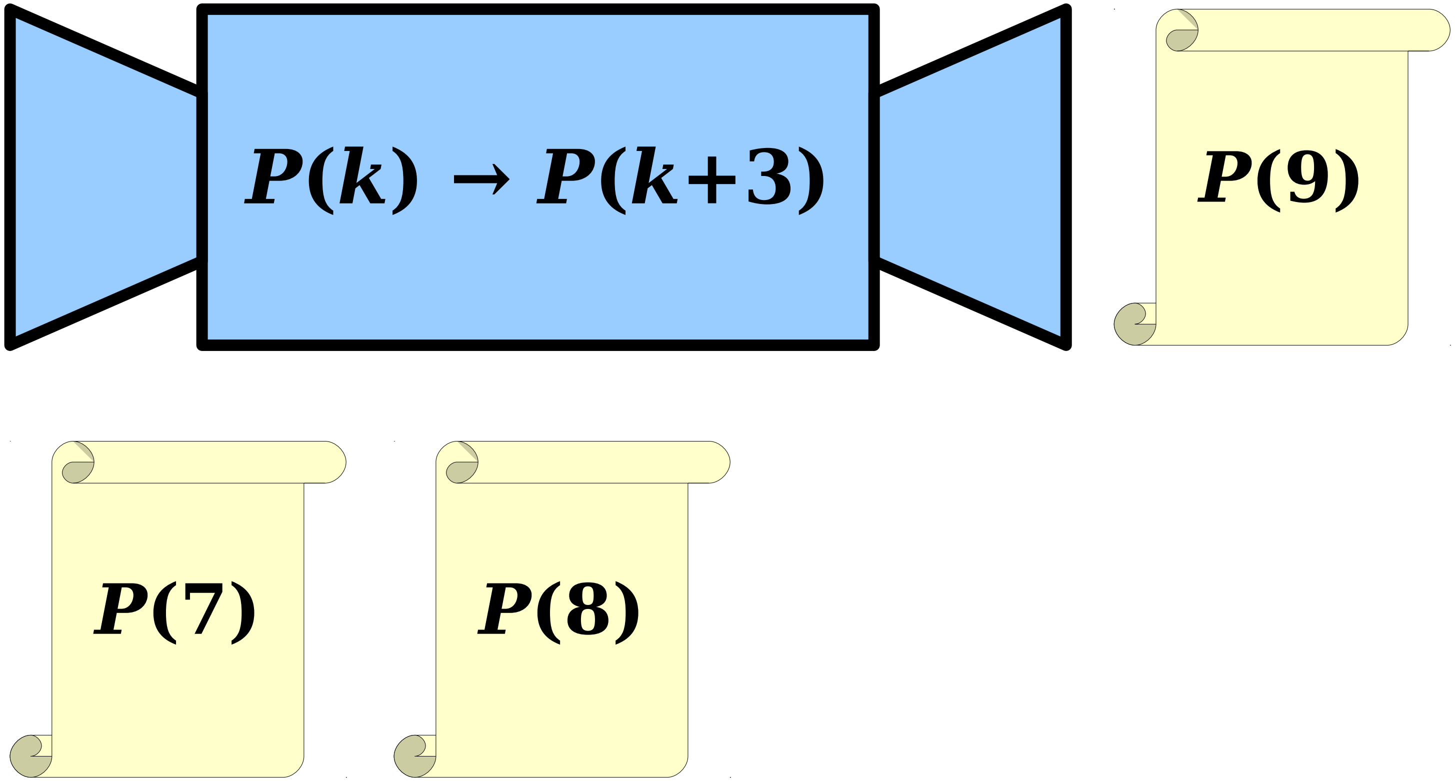

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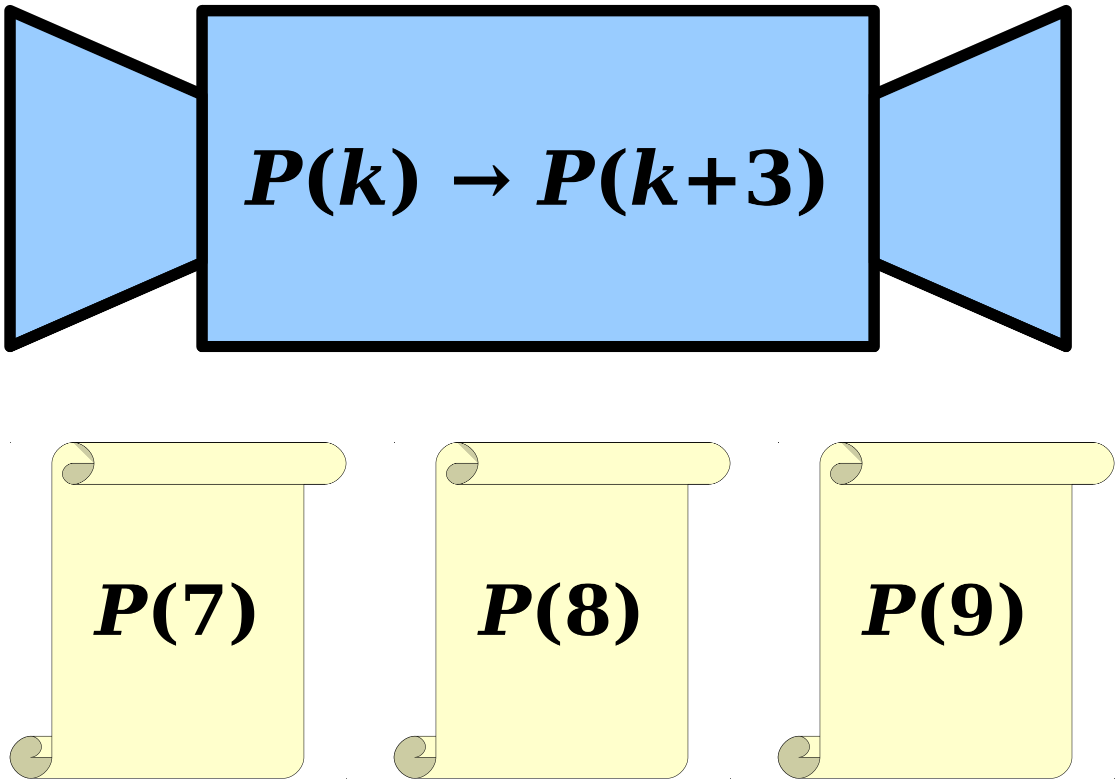

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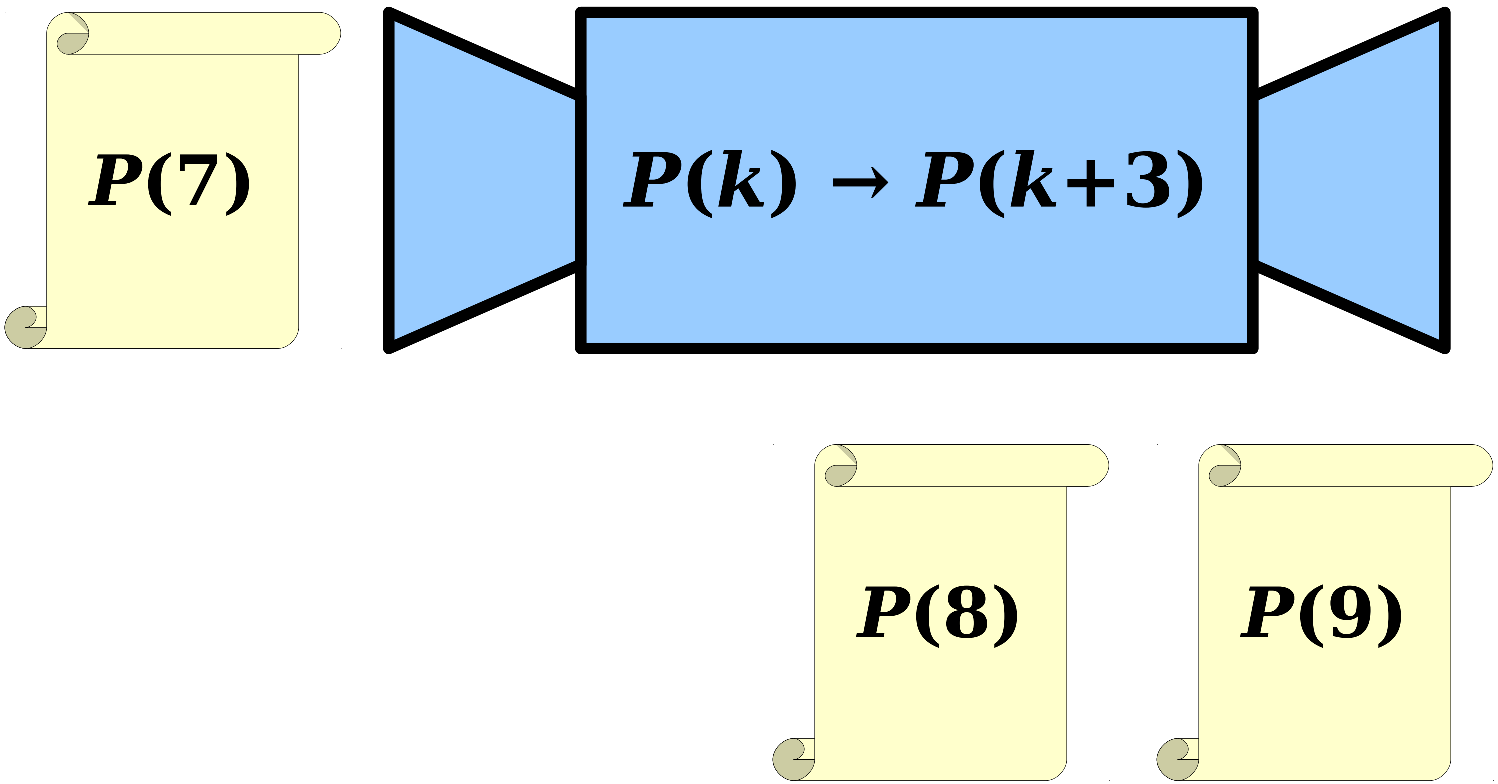

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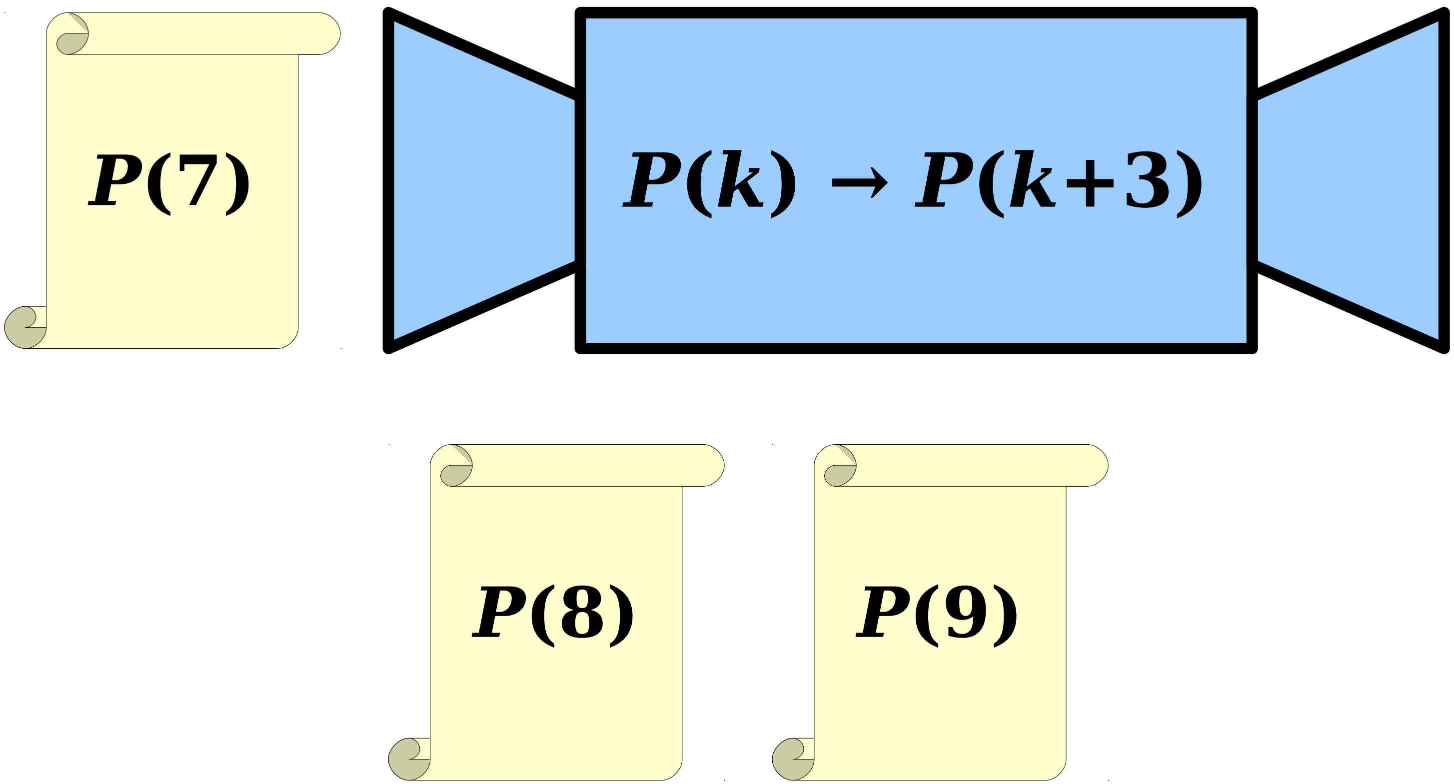

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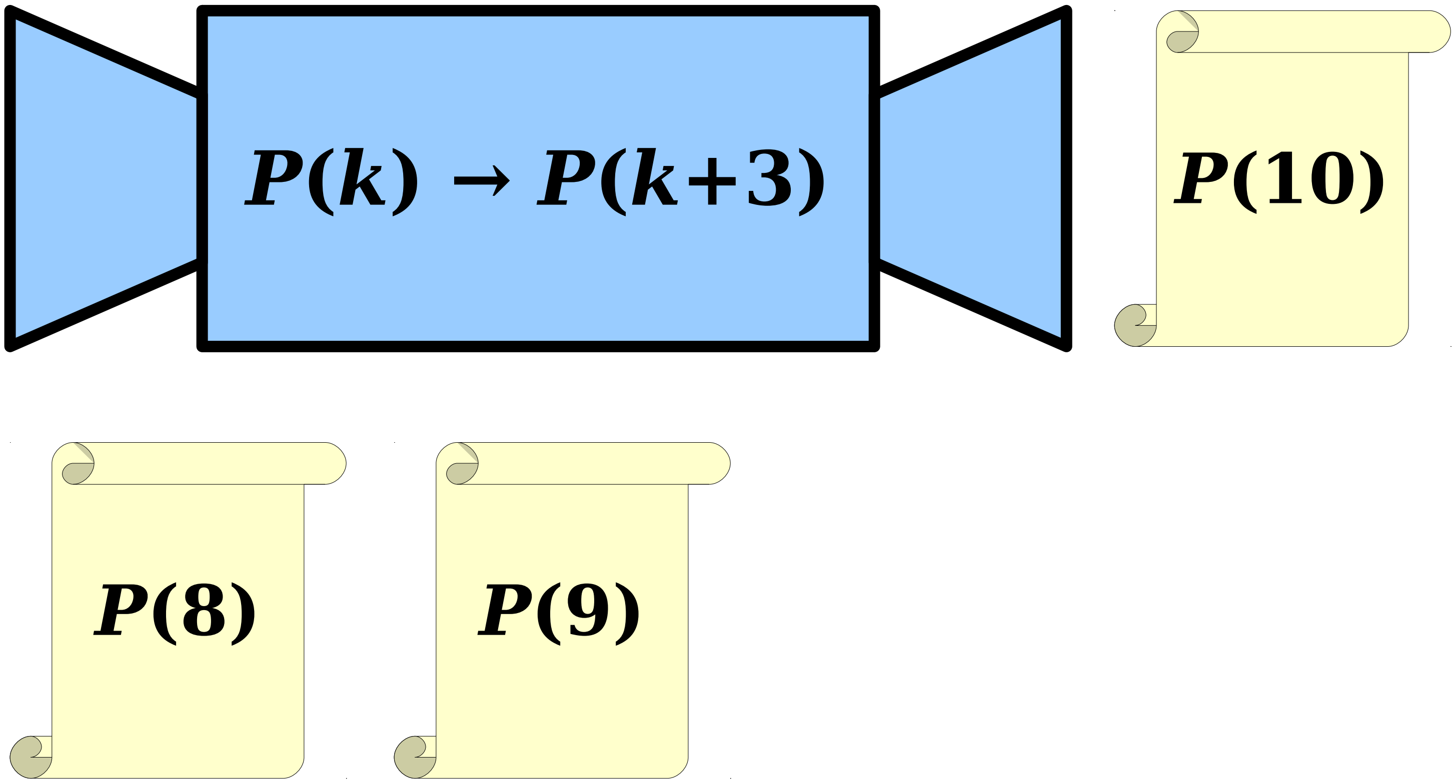

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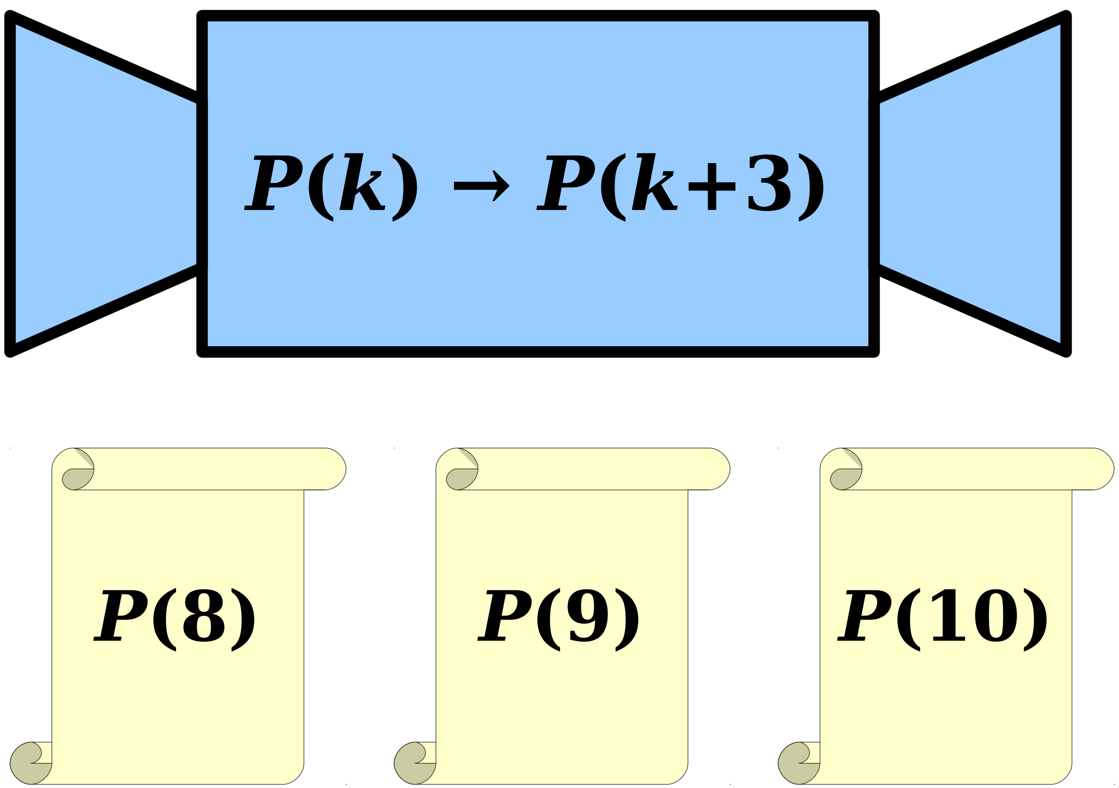

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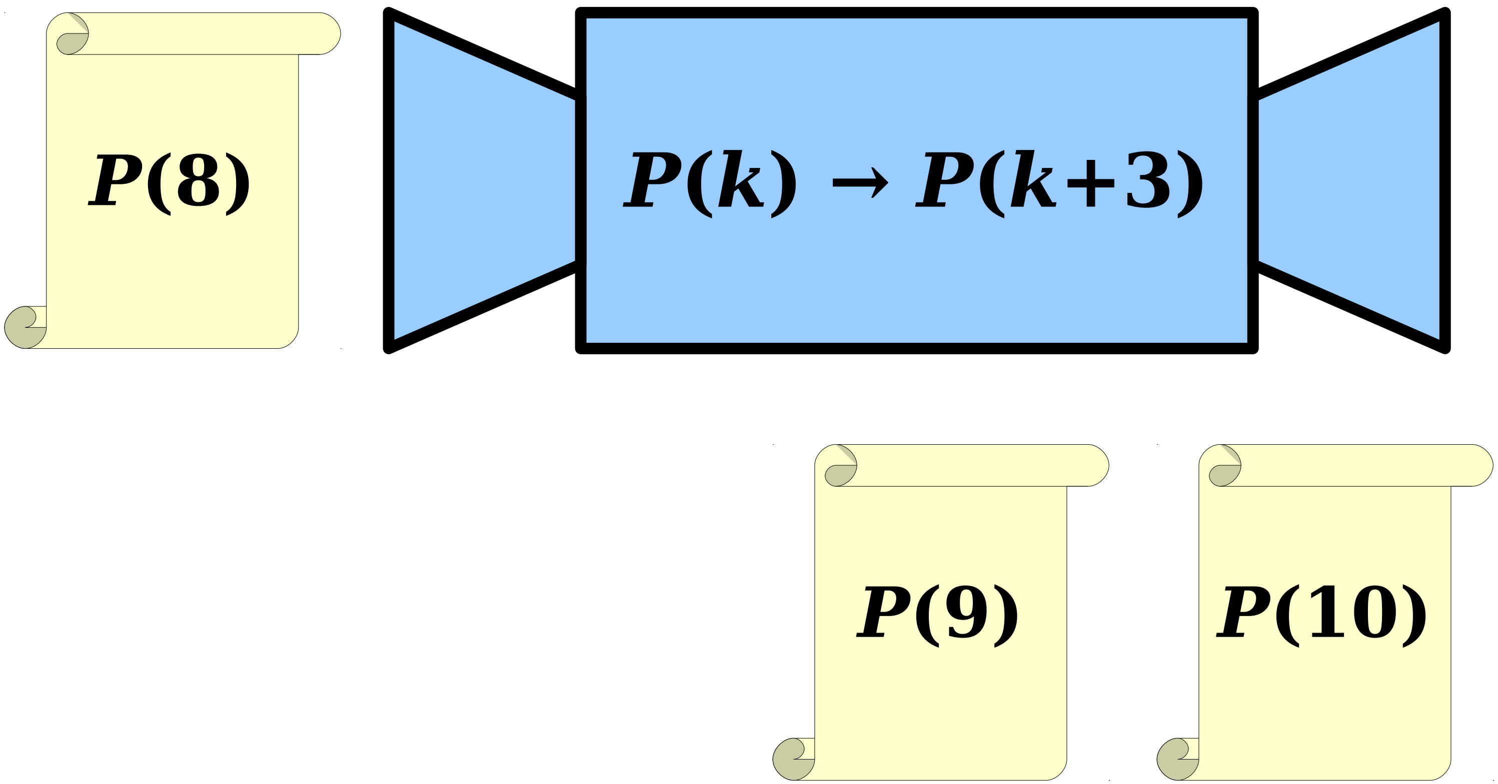

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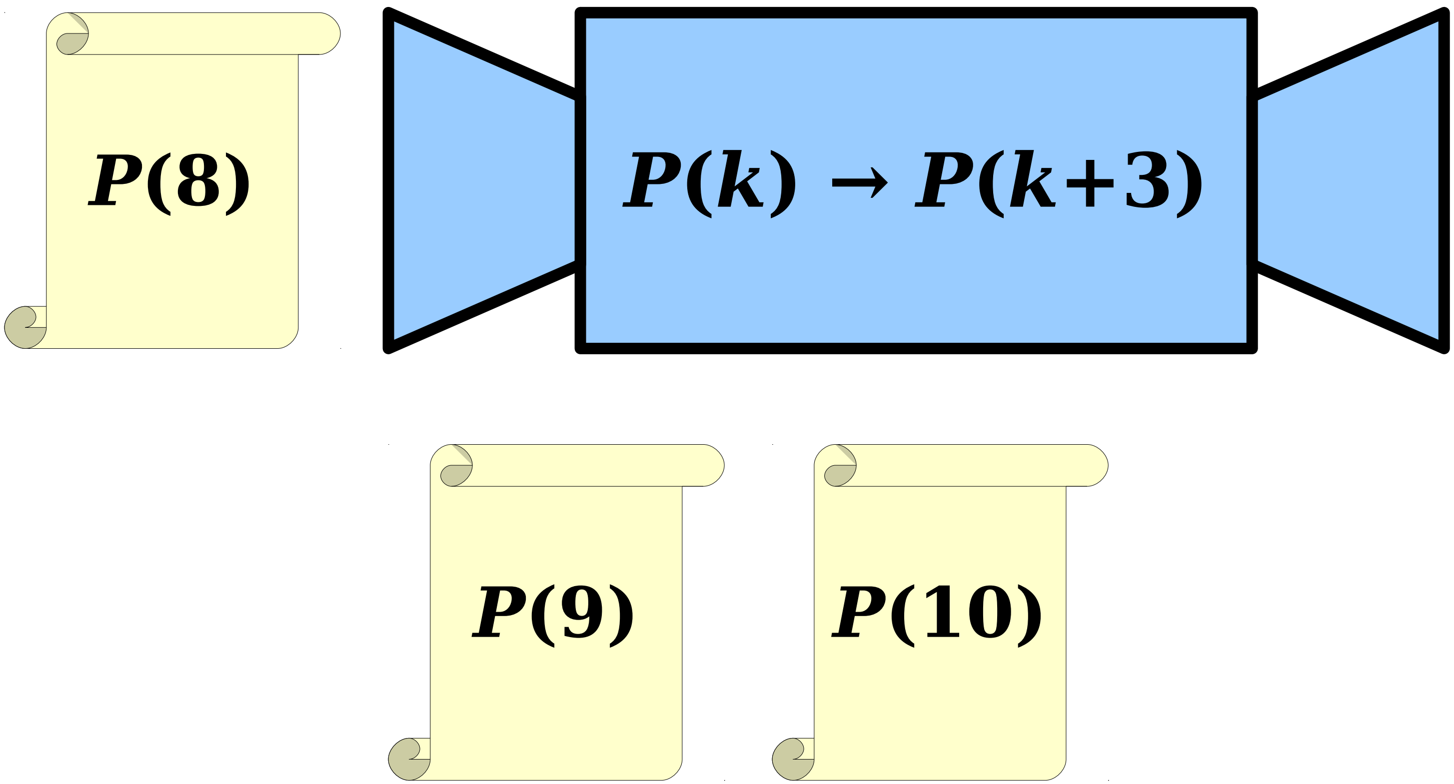

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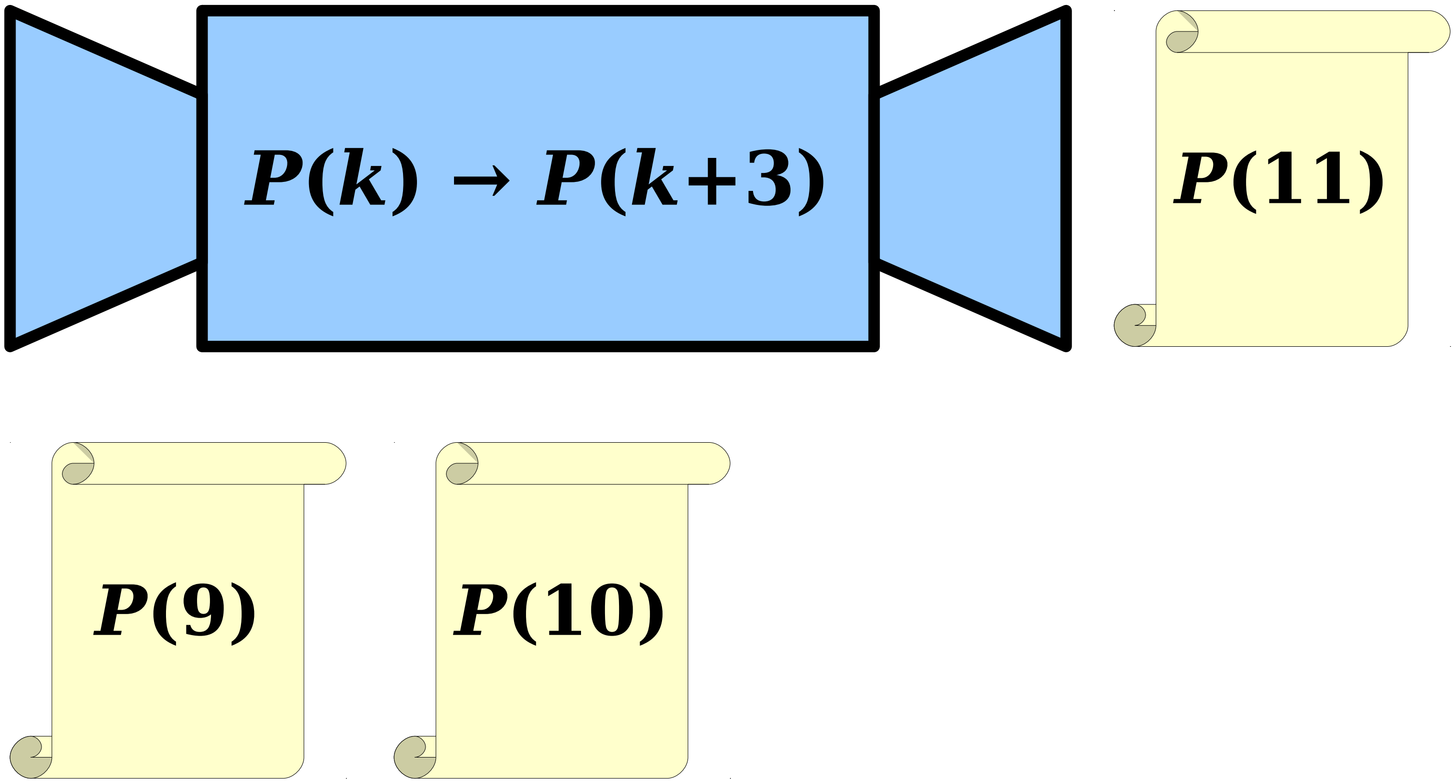

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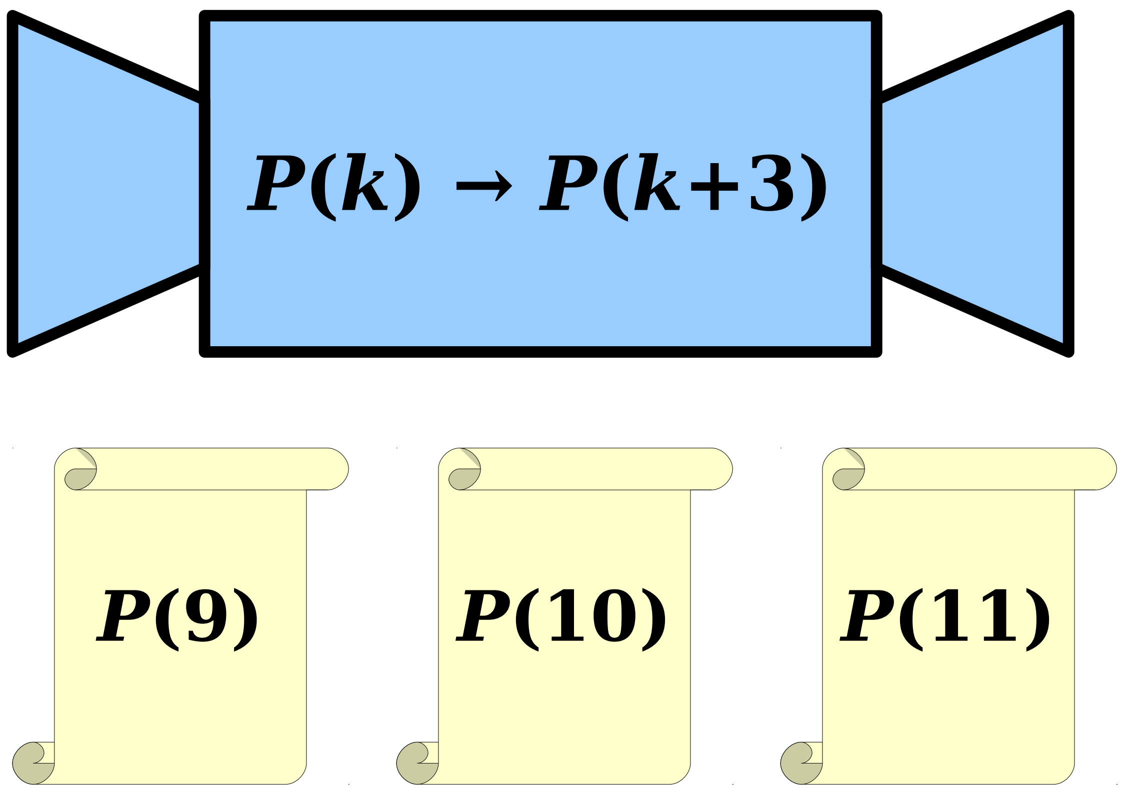

Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- Just be careful to make sure you cover all the numbers you think that you're covering!
  - We won't require that you prove you've covered everything, but it doesn't hurt to double-check!

More on Square Subdivisions

- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on ***Squaring the Square***.

Complete Induction

Guess what!?

It’s time for

Mathematical

Calesthenics!

It’s time for

Mathematicalesthenics!

This is kinda like $P(0)$.

If you are the ***leftmost*** person in your row, stand up right now.

Everyone else: stand up as soon as the person to your left in your row stands up.

This is kinda like $P(k) \to P(k+1)$.

Everyone, please be seated.

Let’s do this again… with a twist!

This is kinda
like $P(0)$.

If you are the ***leftmost*** person
in your row, stand up right now.

Everyone else: stand up as soon as
everyone left of you in your row stands up.

What sort of sorcery
is this?

Please be seated.

You all did a great job!

Let P be some predicate. The ***principle of complete induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

for any $k \in \mathbb{N}$, if $P(0)$, $P(1)$, …, and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

Mathematical Induction

- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

Complete Induction

- You can write proofs using the principle of ***complete*** induction as follows:
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  - Pick an arbitrary $k \in \mathbb{N}$ and assume that ***$P(0)$, $P(1)$, $P(2)$, …, and $P(k)$*** are all true.
  - Prove $P(k+1)$.
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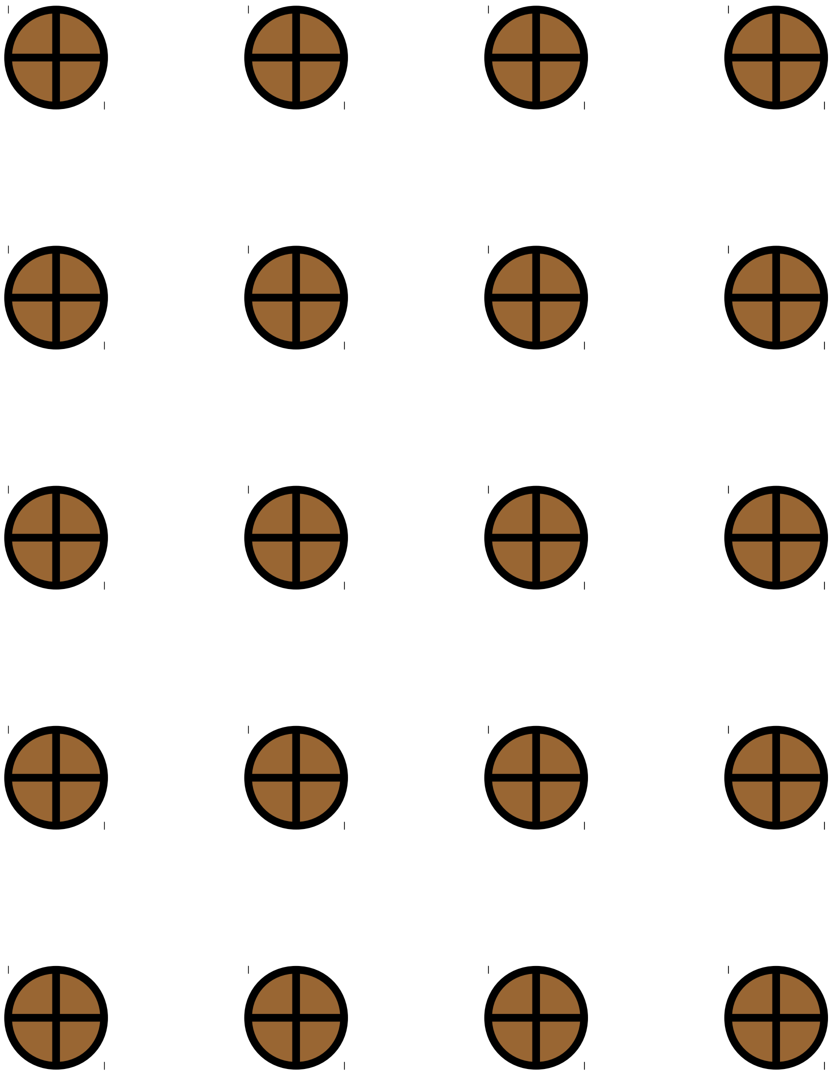
A Motivating Example: ***Rat Mazes***

Rat Mazes

- Suppose you want to make a rat maze consisting of an $n \times m$ grid of pegs with slats between them.



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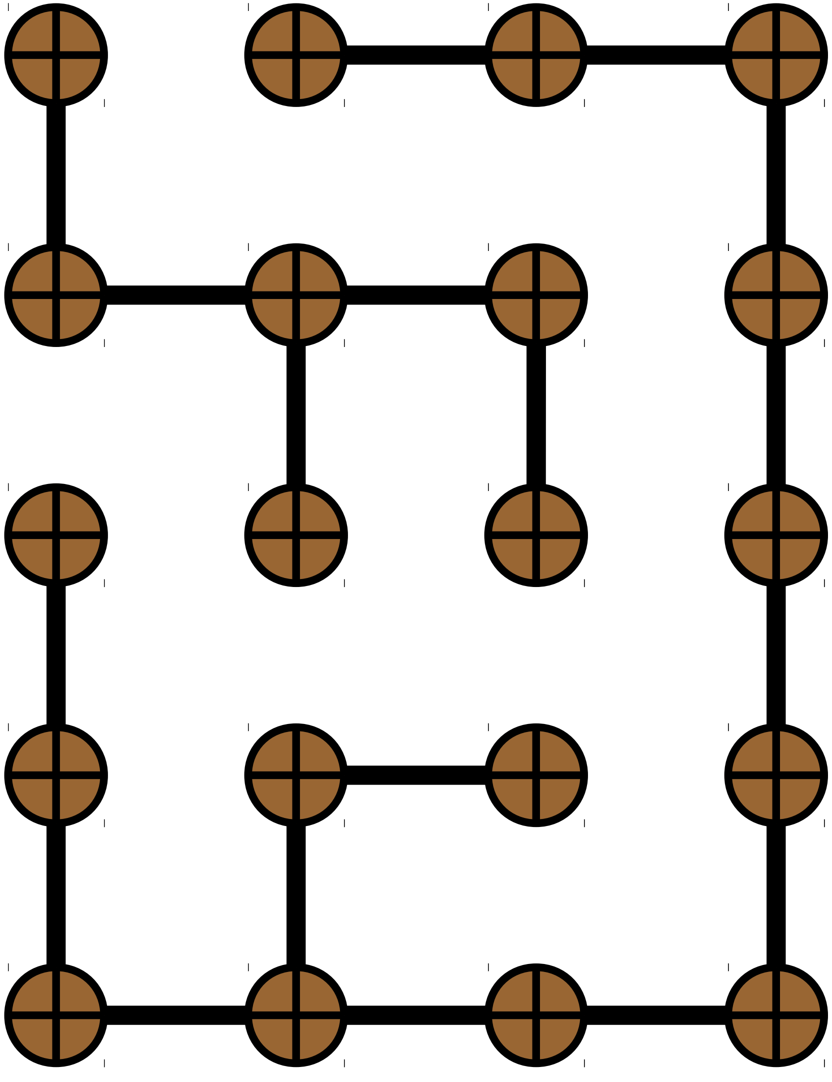
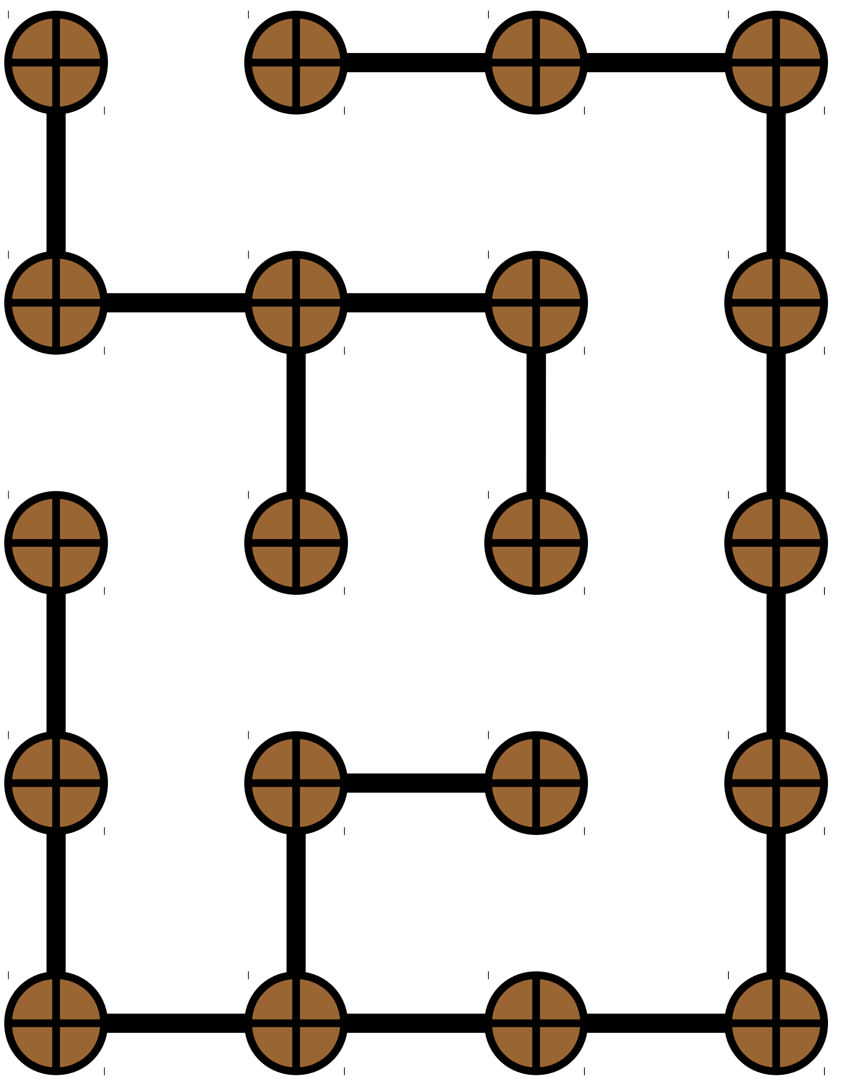


Rat Mazes

- Suppose you want to make a rat maze consisting of an $n \times m$ grid of pegs with slats between them.
- The maze should have these properties:
  - There is one entrance and one exit in the border.
  - Every spot in the maze is reachable from every other spot.
  - There is exactly one path from each spot in the maze to each other spot.

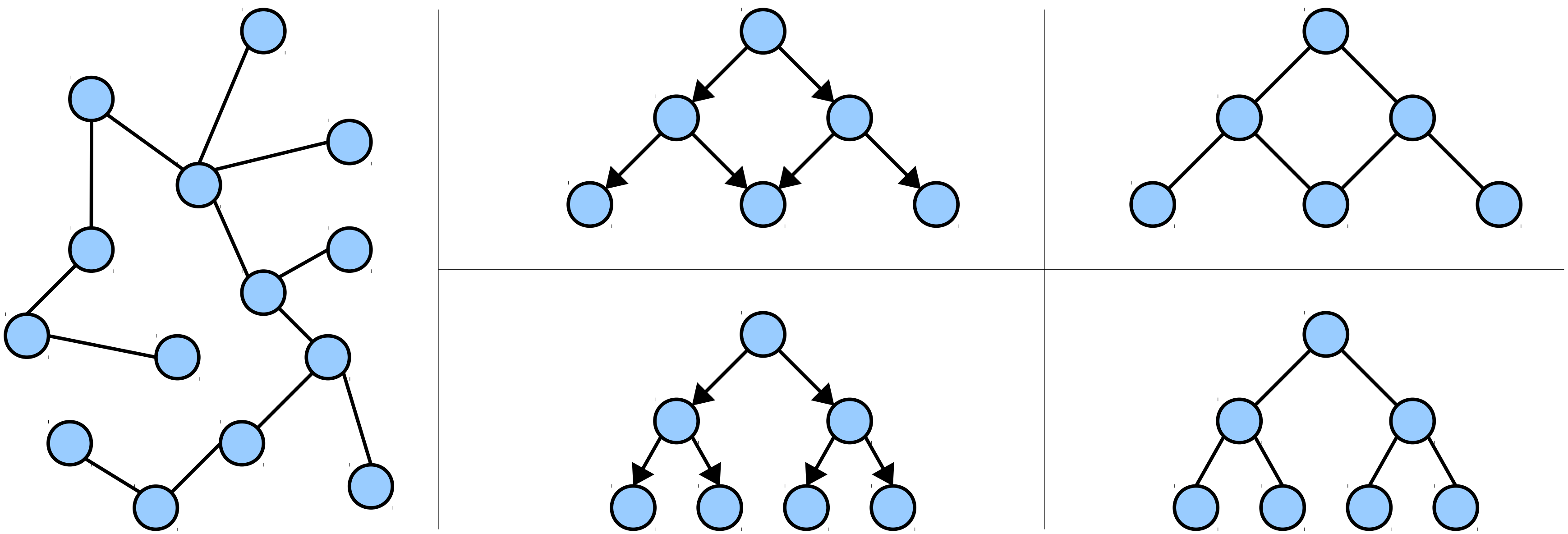


Question: If you have an $n \times m$ grid of pegs, how many slats do you need to make?

A Special Type of Graph: ***Trees***

A ***tree*** is a connected, nonempty graph with no simple cycles.

According to the above definition of trees, how many of these graphs are trees?



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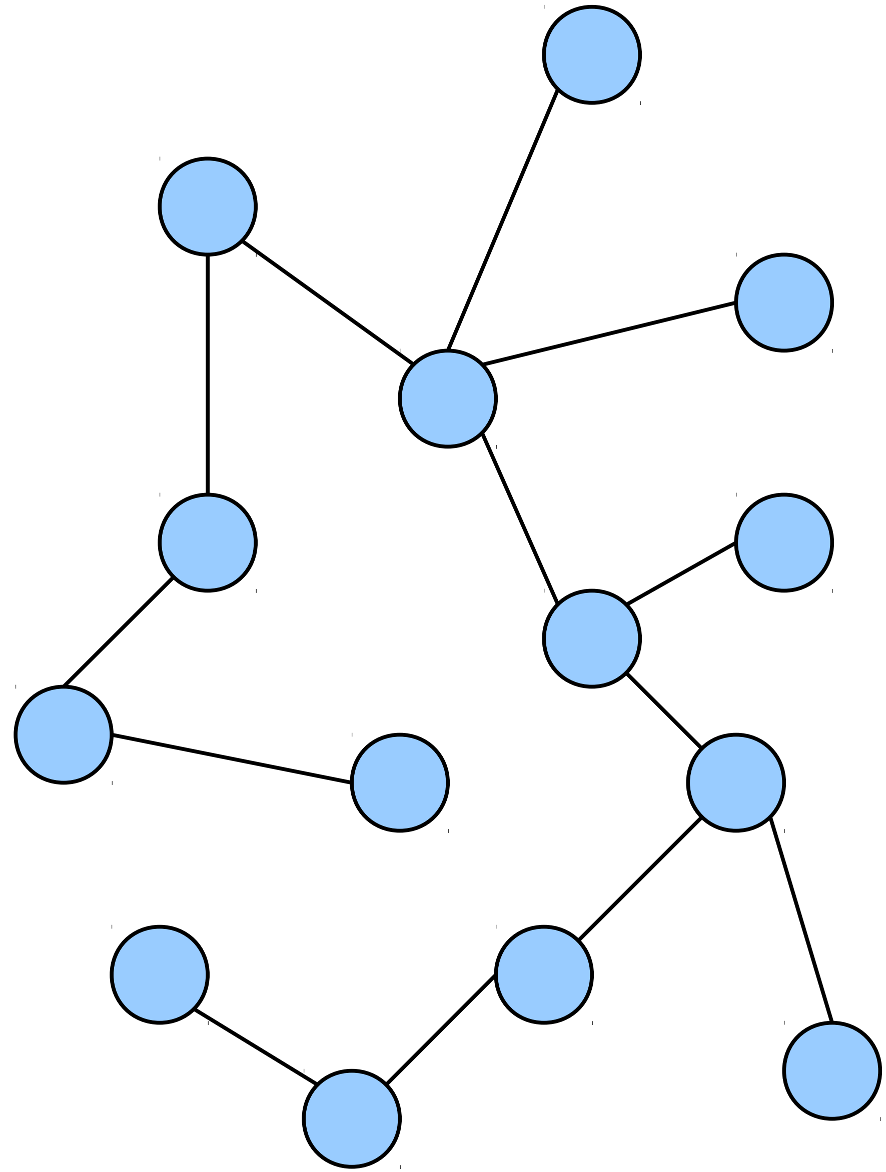

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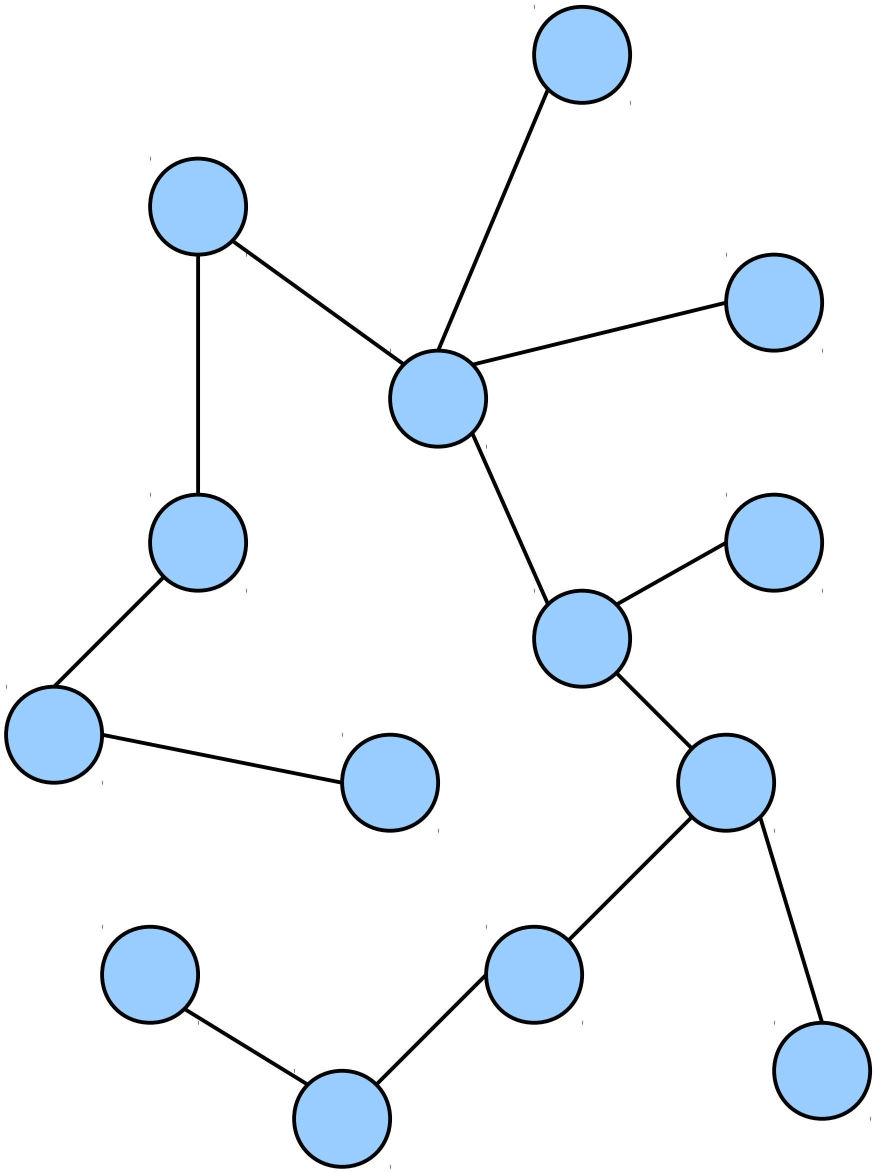
- A ***tree*** is a connected, nonempty graph with no simple cycles.
- Trees have tons of nice properties:
  - They're ***maximally acyclic*** (adding any missing edge creates a simple cycle)



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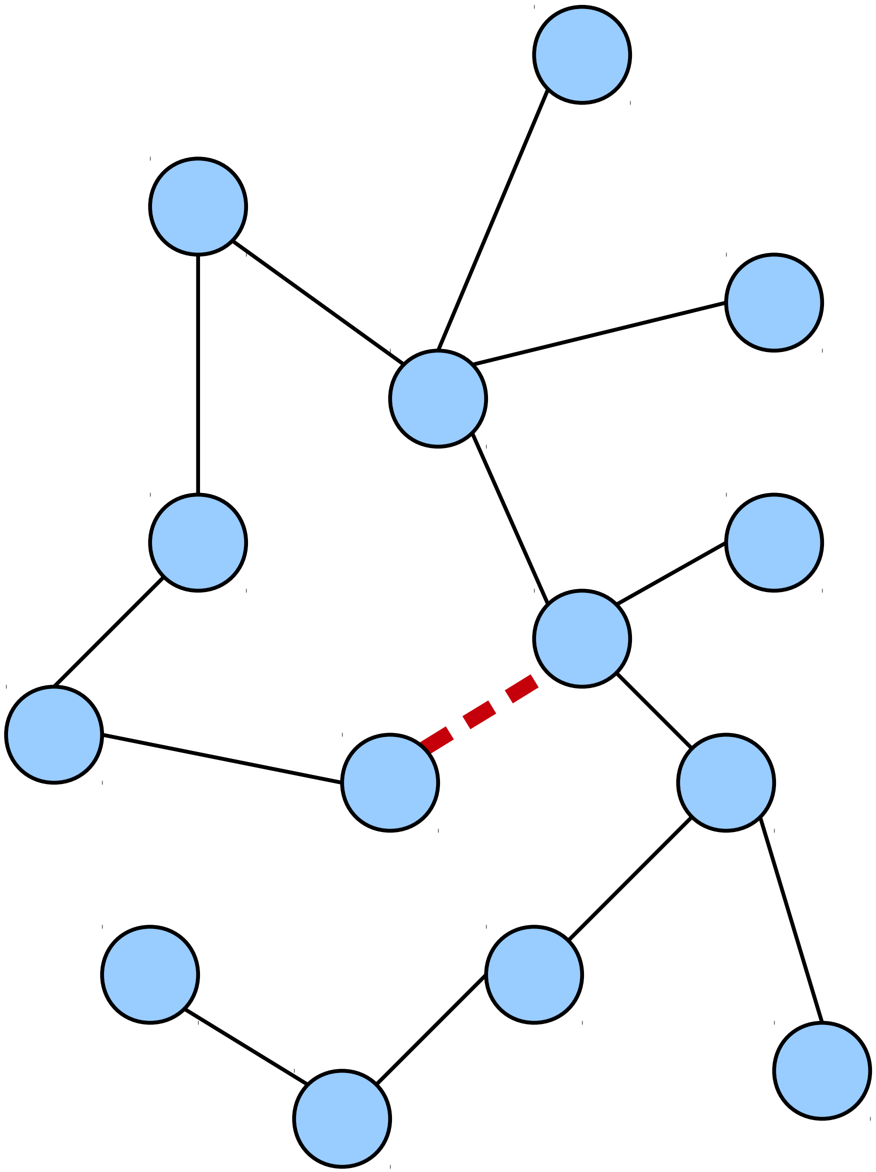

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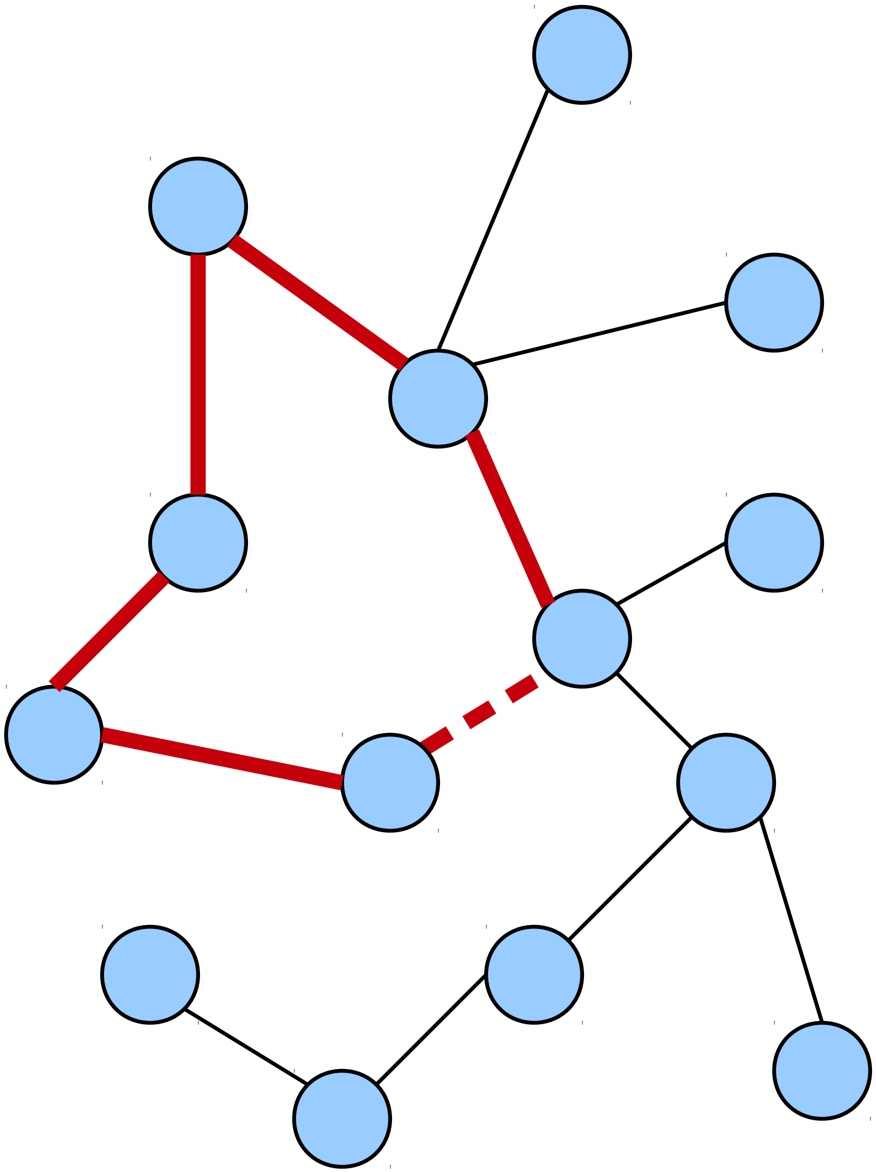

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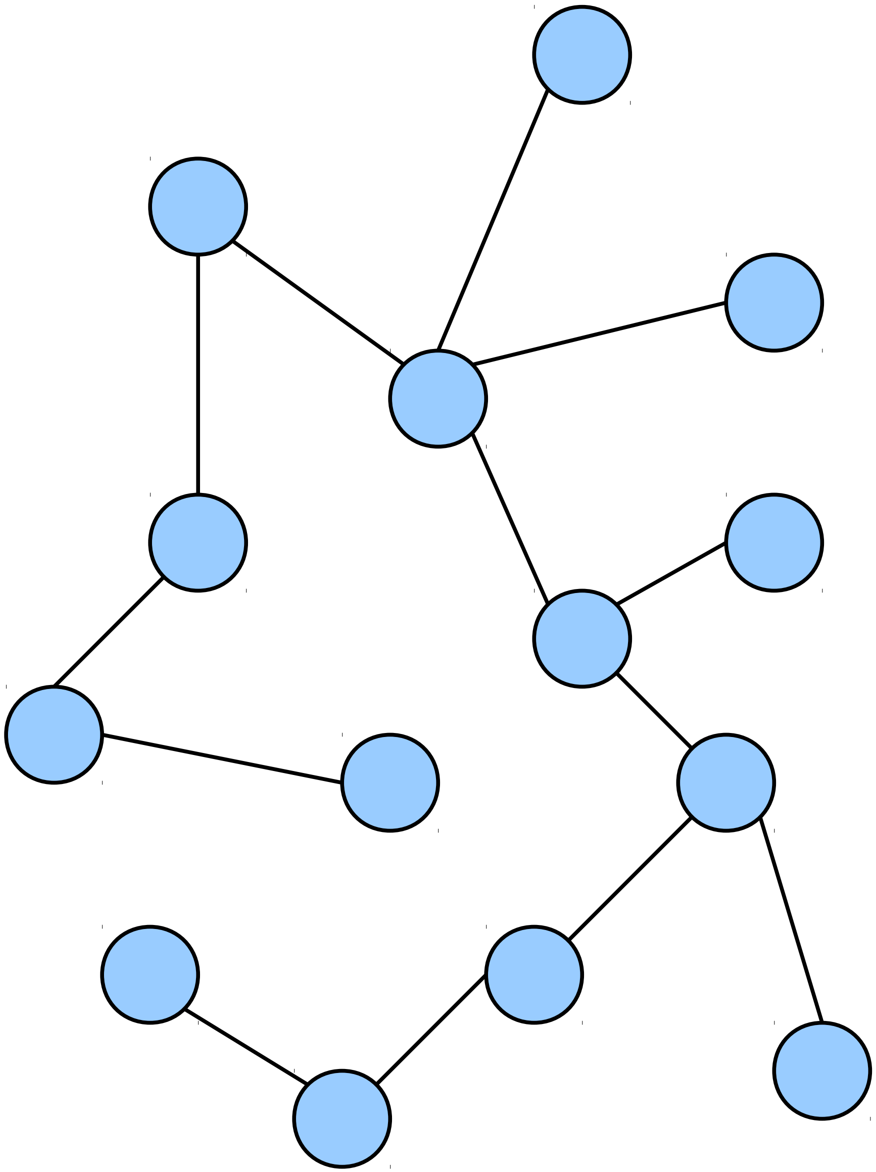

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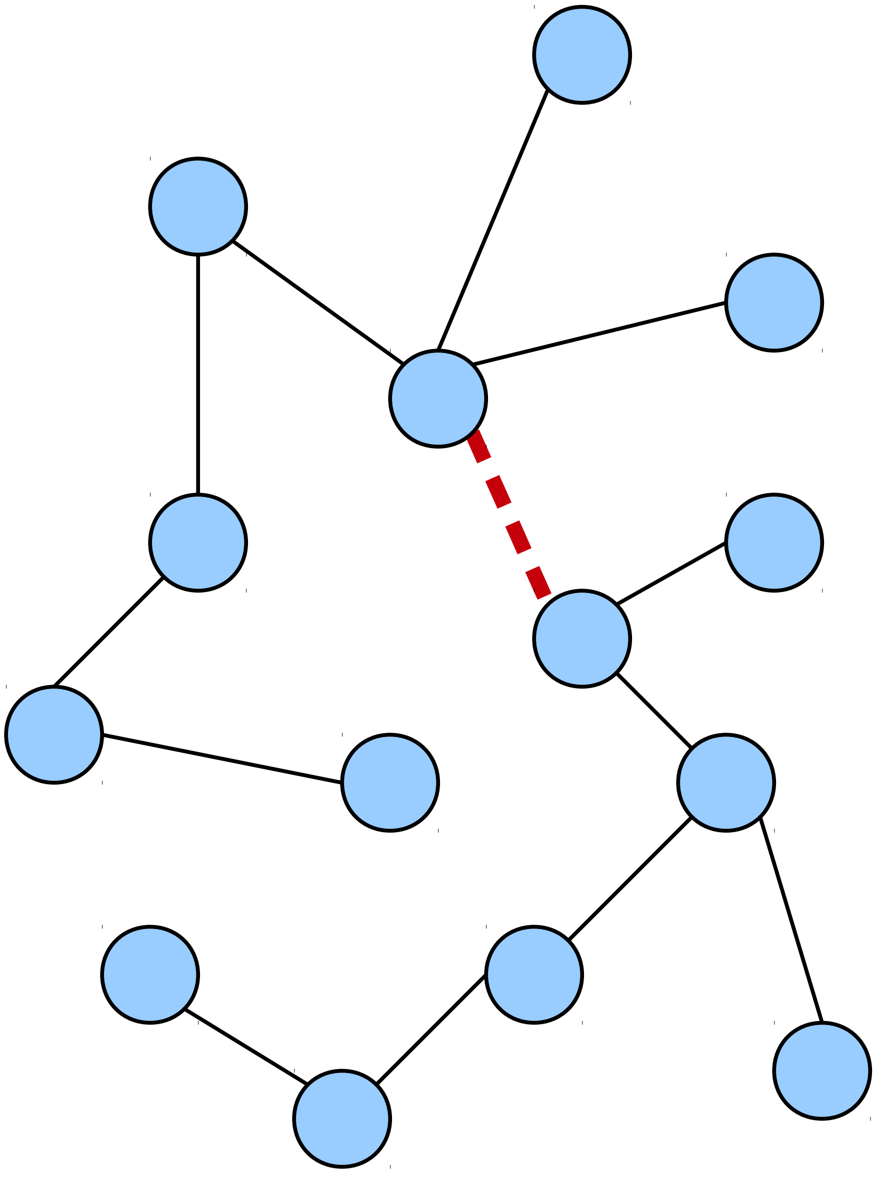

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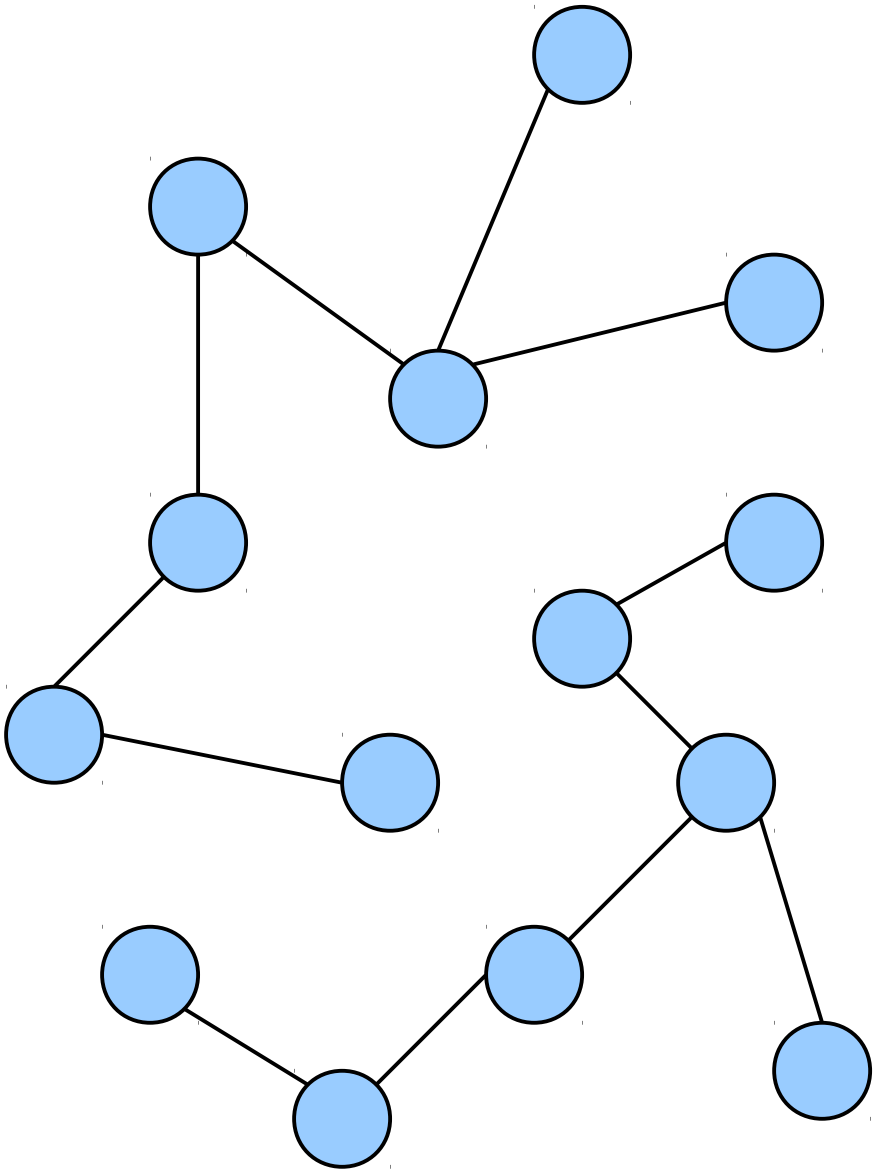

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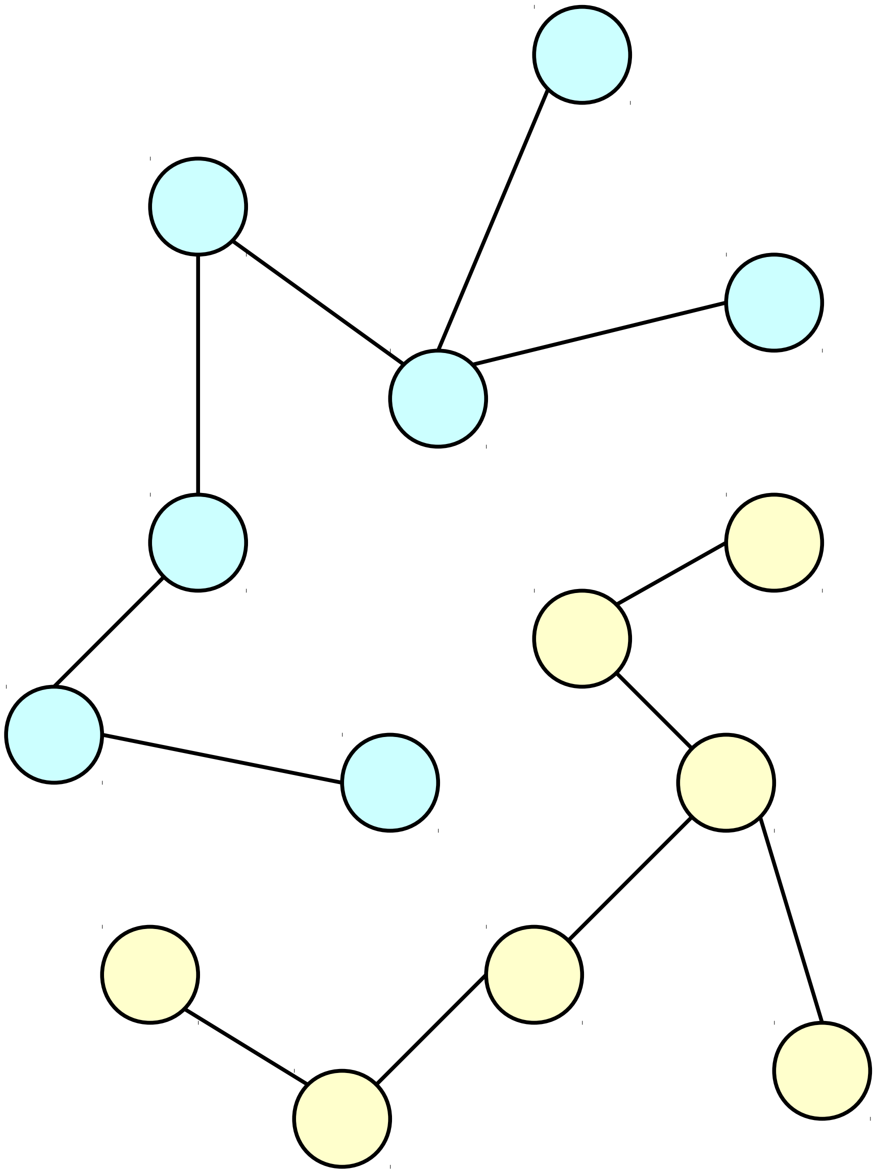

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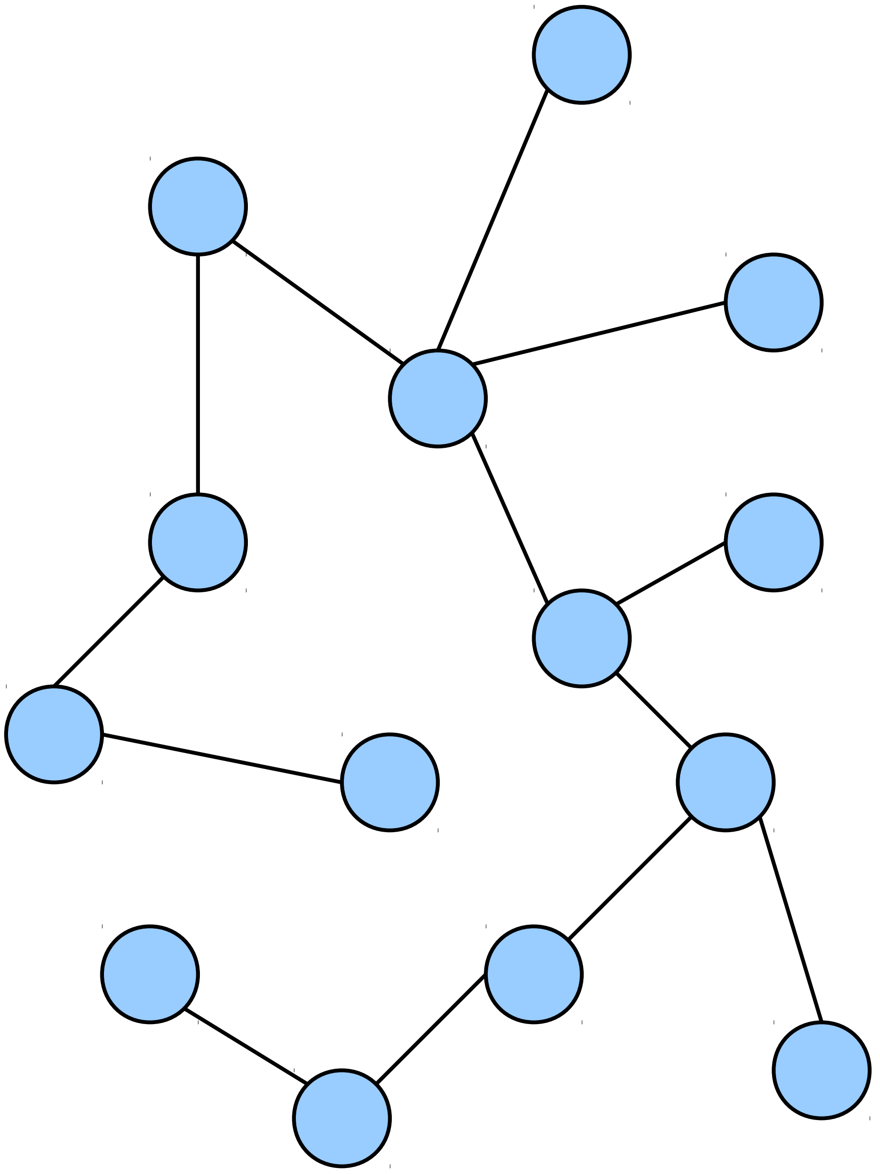
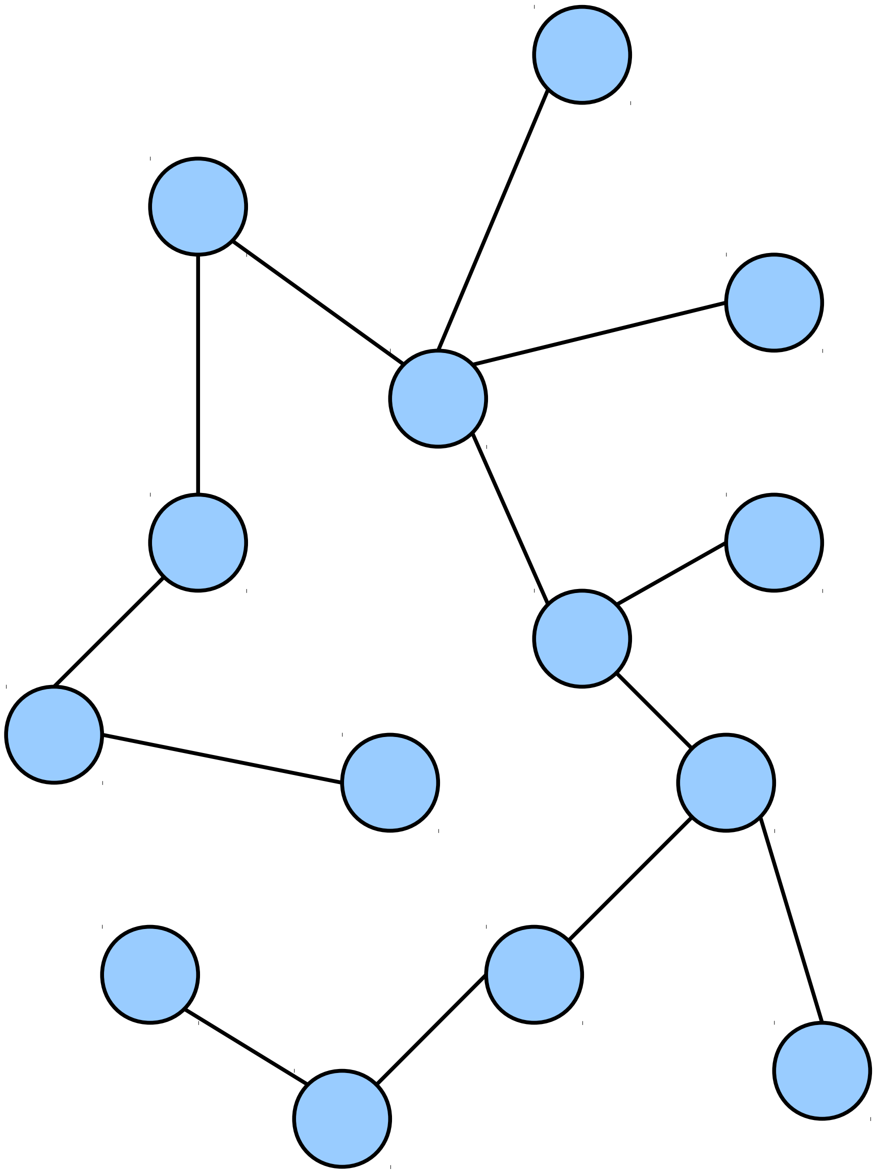
- Proofs of these results are in the course reader if you're interested. They're also great exercises.



Trees

- ***Theorem:*** If T is a tree with at least two nodes, then deleting any edge from T splits T into two nonempty trees T_1 and T_2.
- ***Proof:*** Left as an exercise to the reader. ☺



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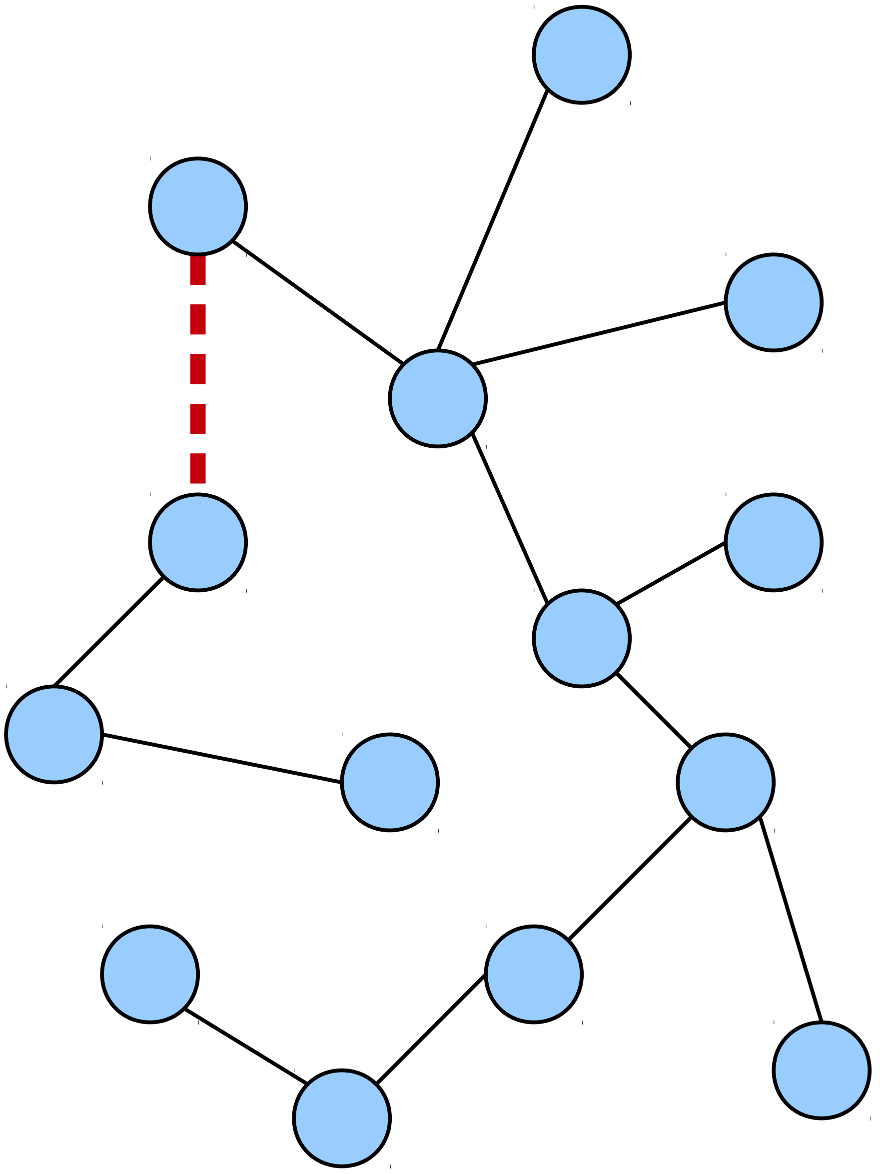

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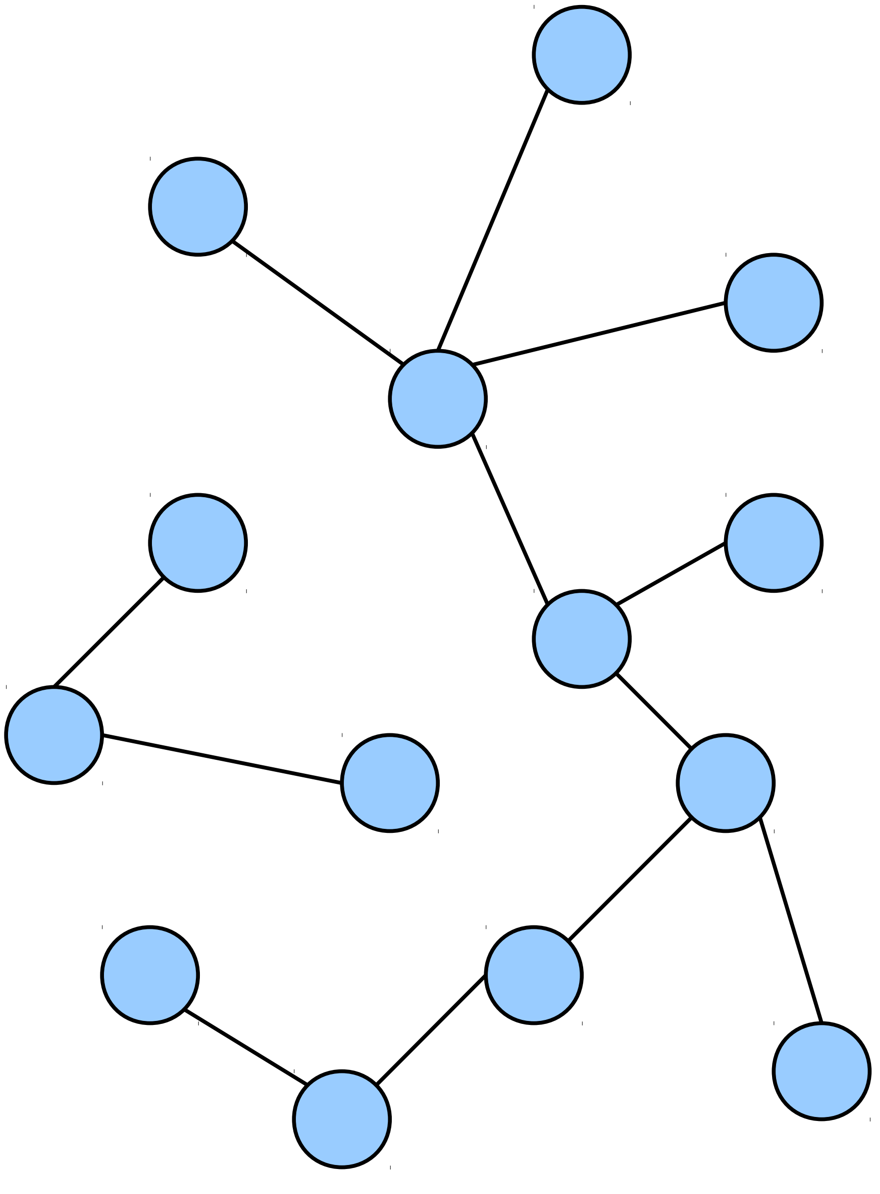

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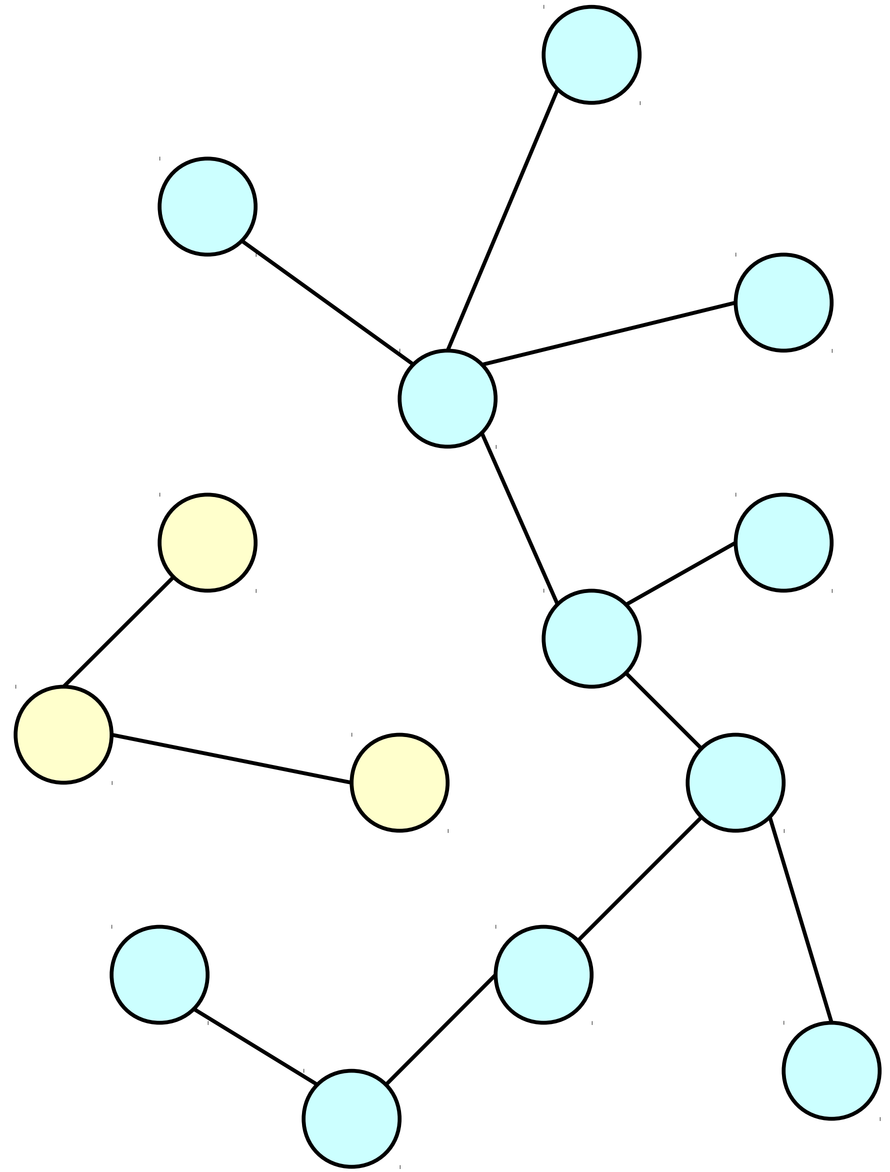

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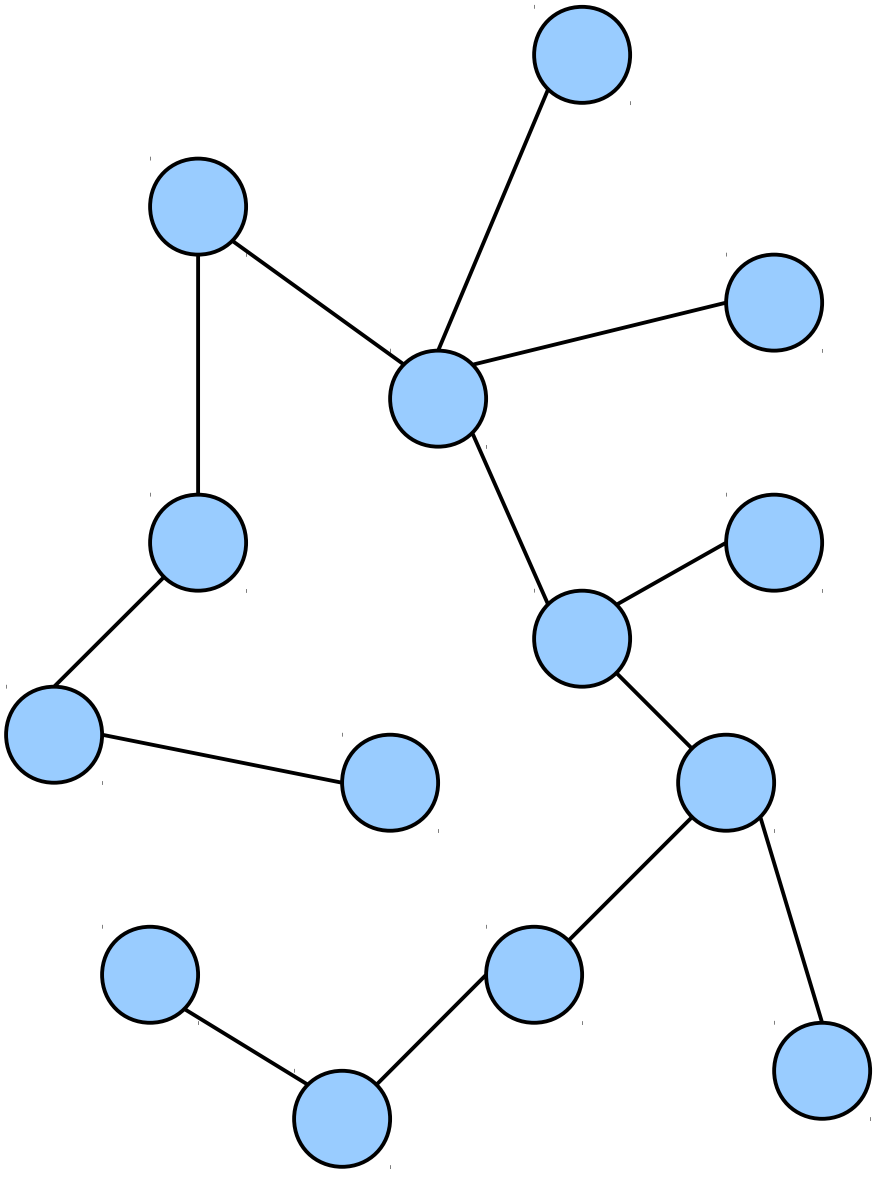

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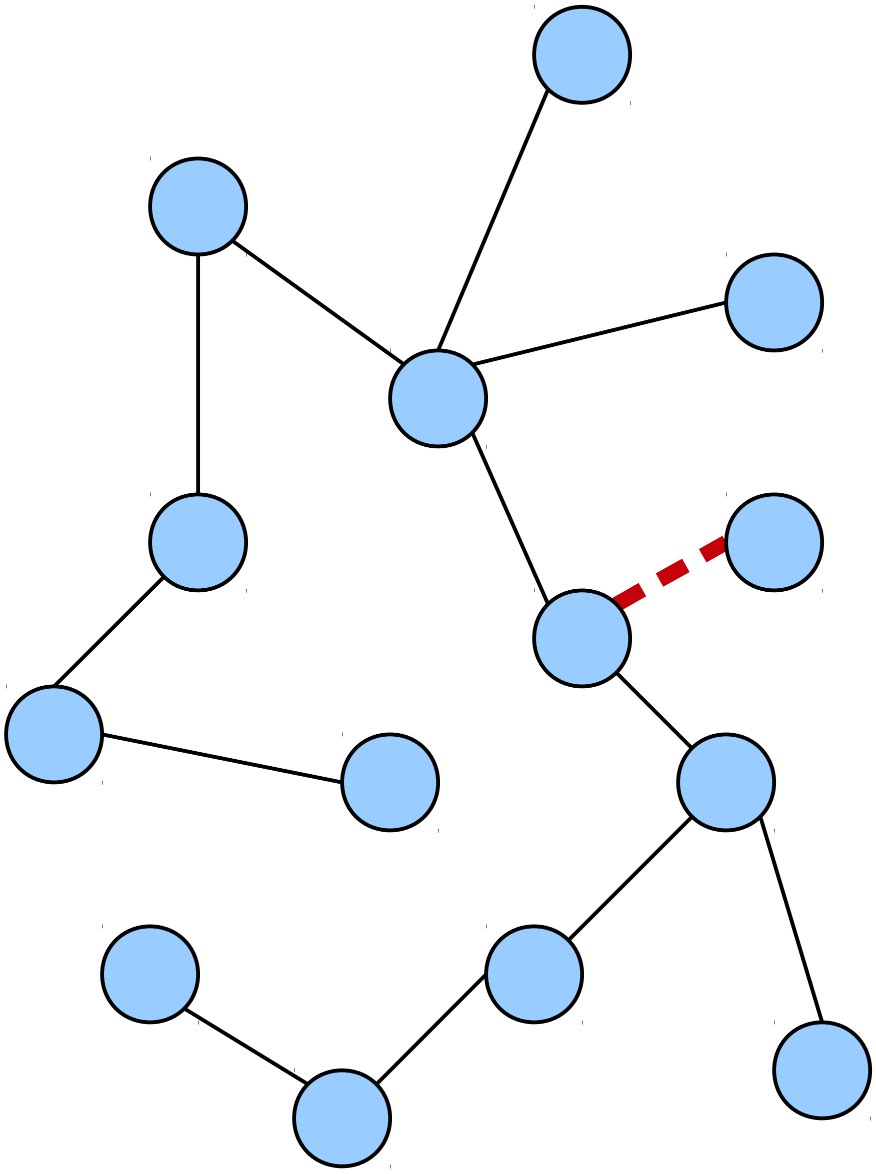

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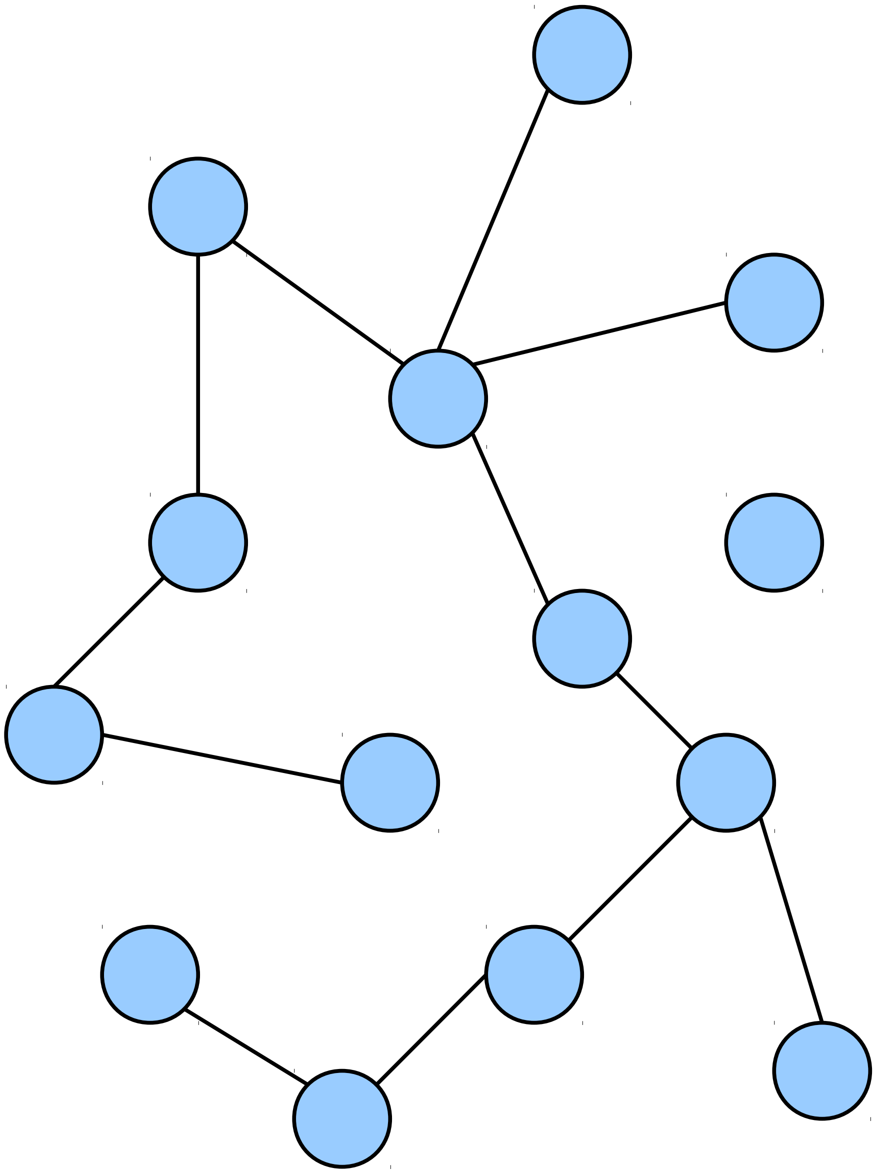

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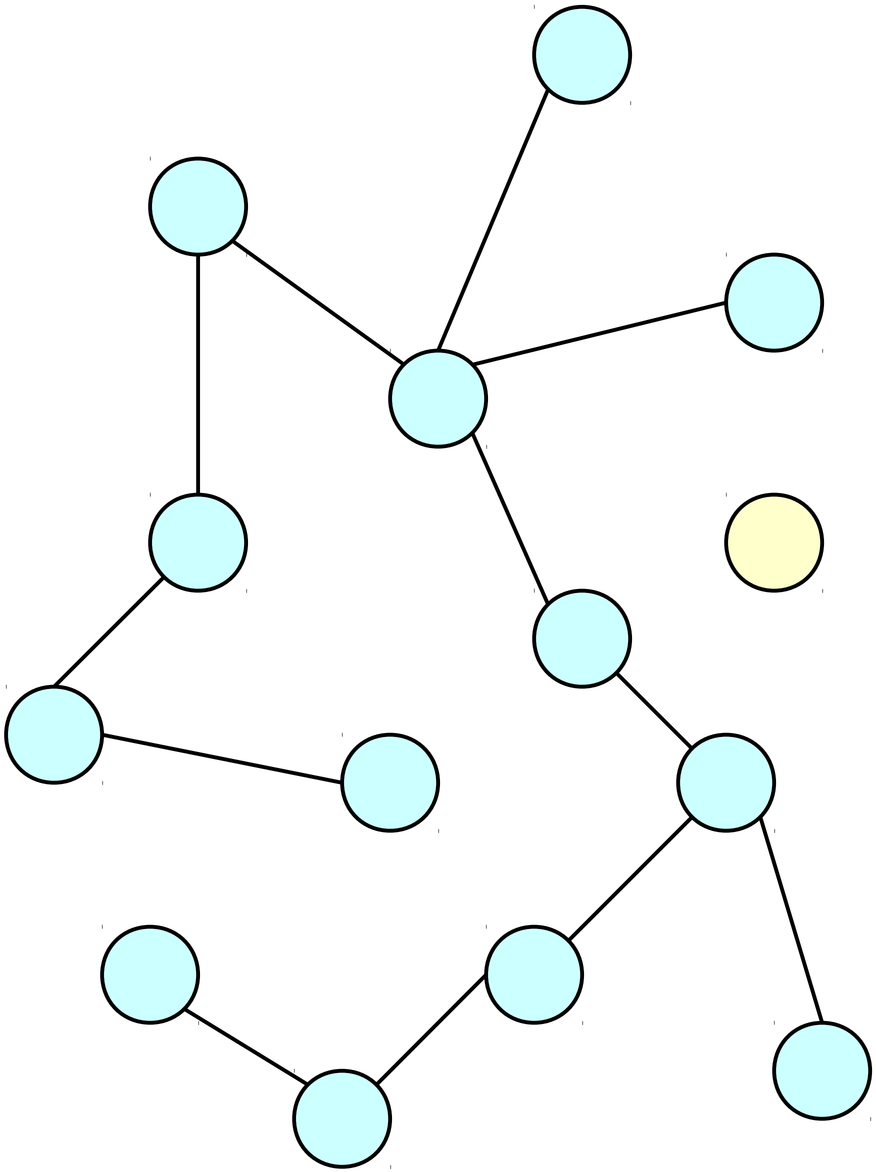

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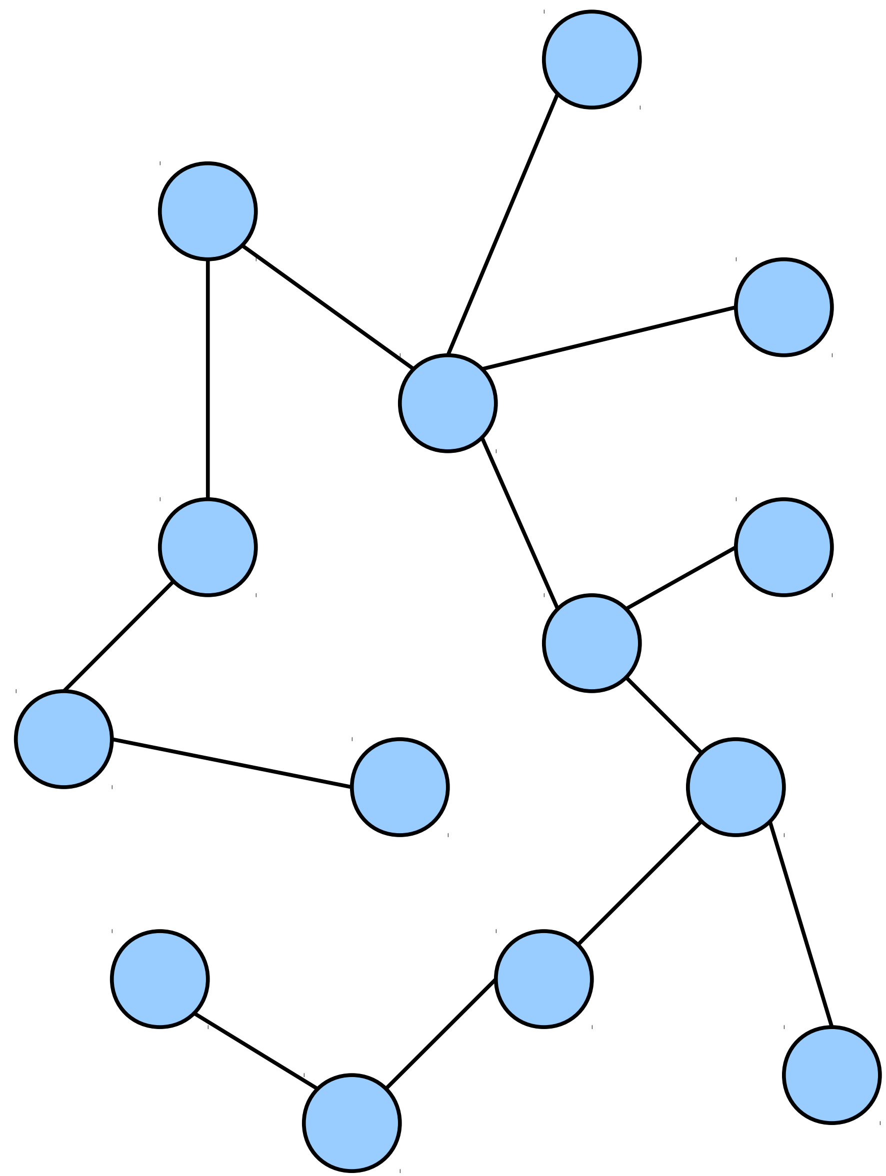

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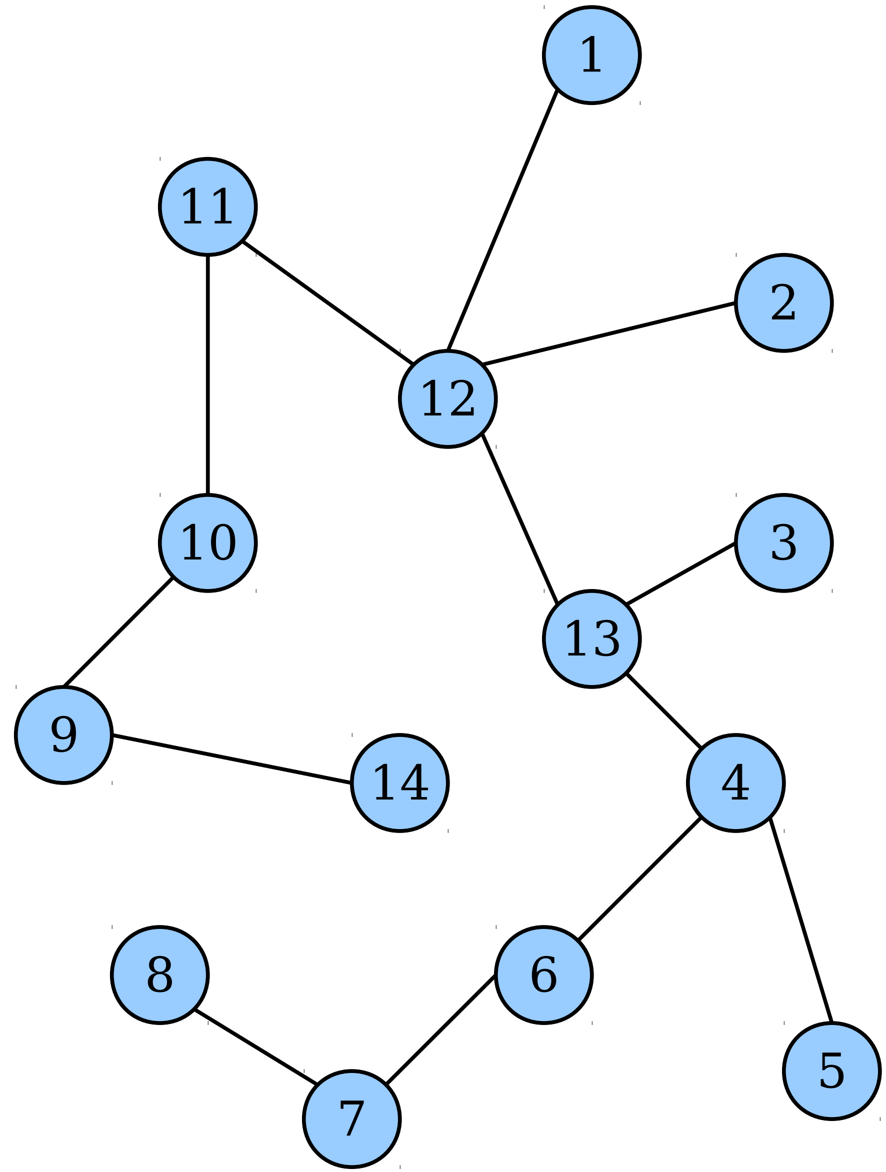

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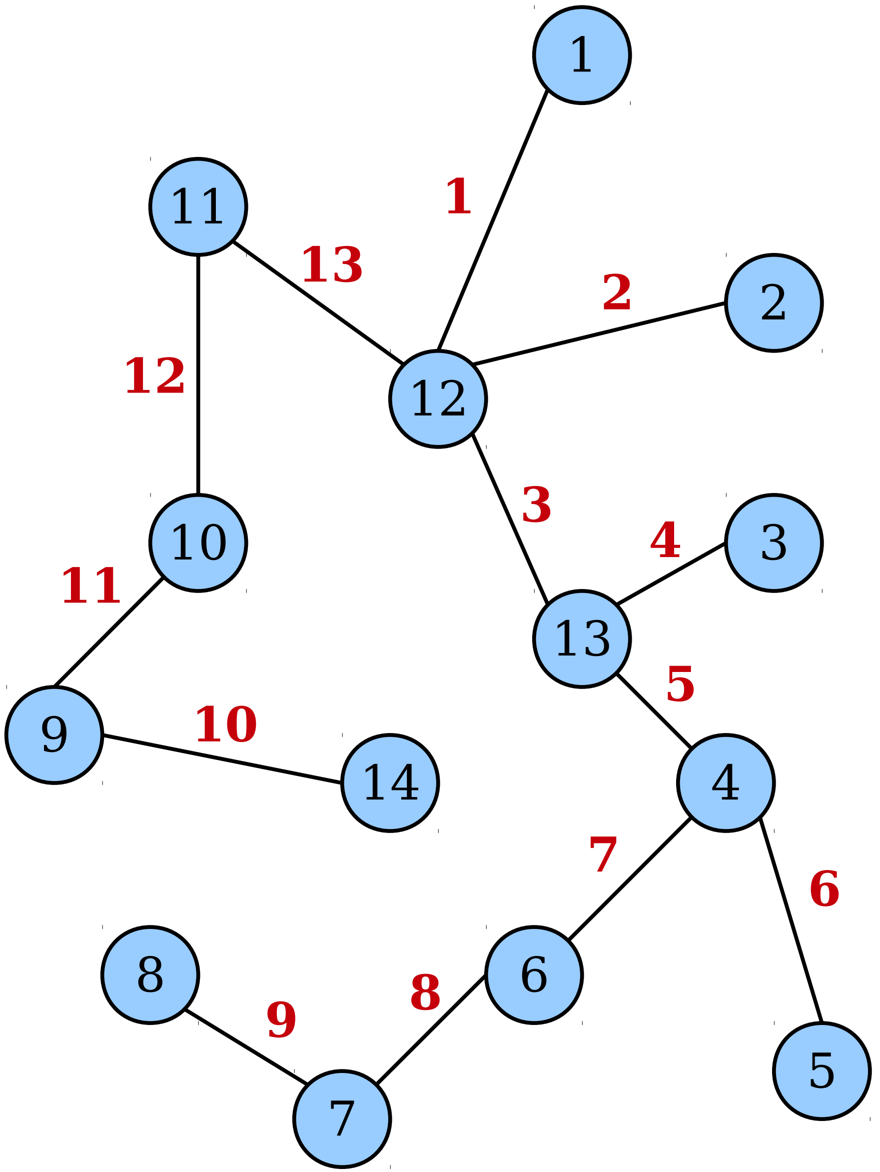
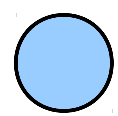


Our Base Case



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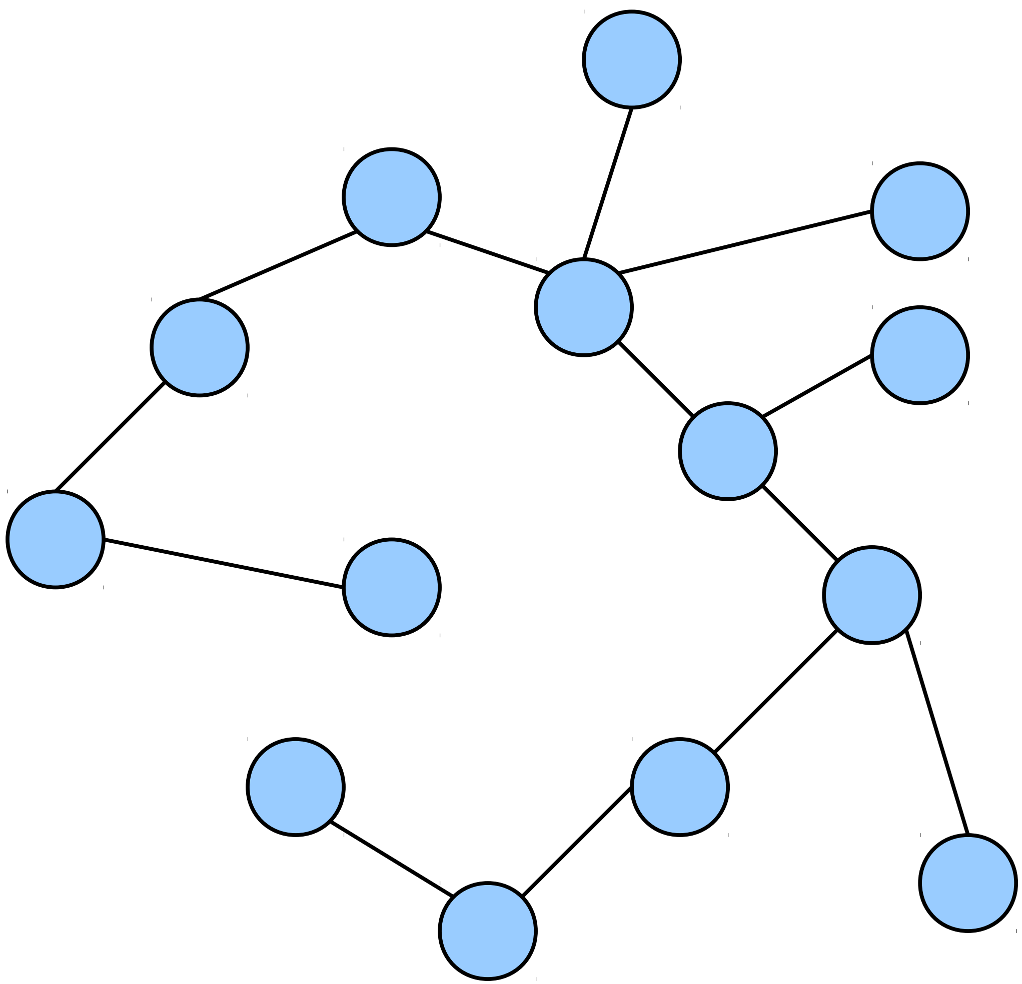

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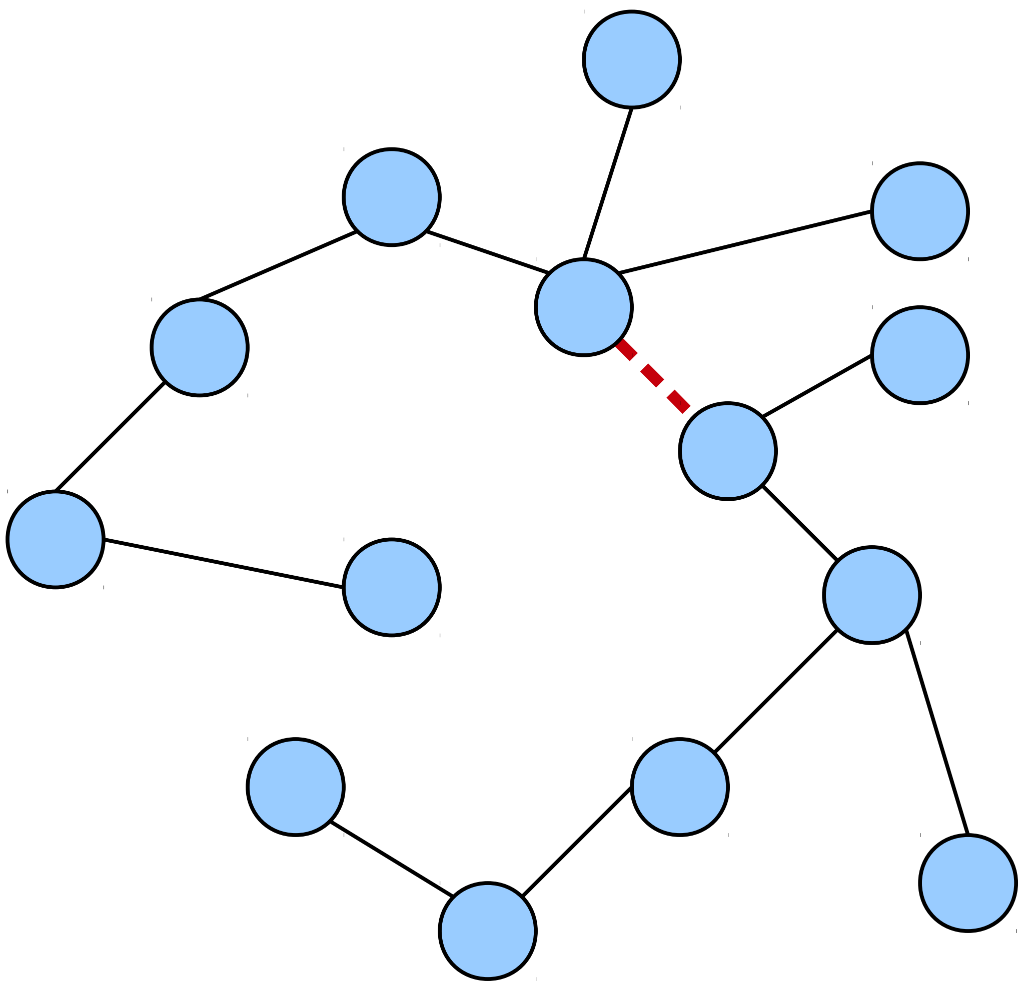

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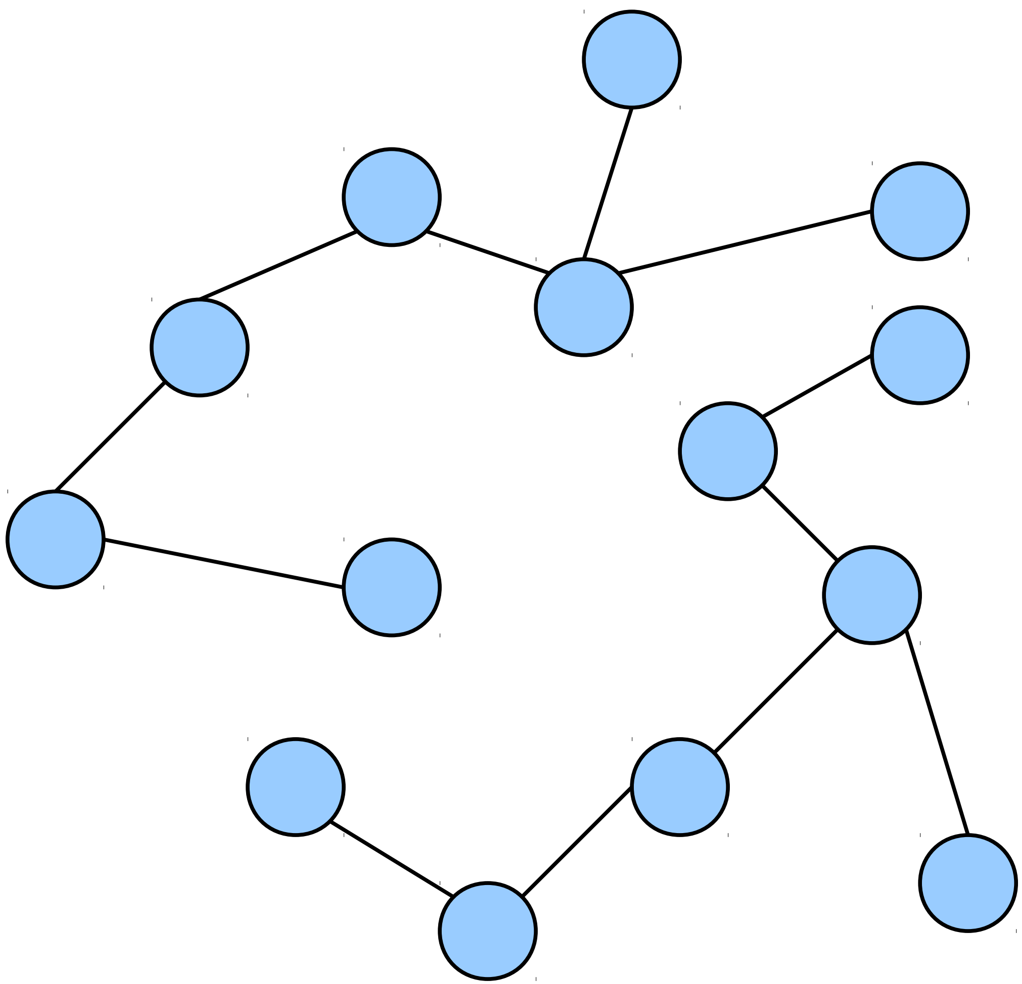

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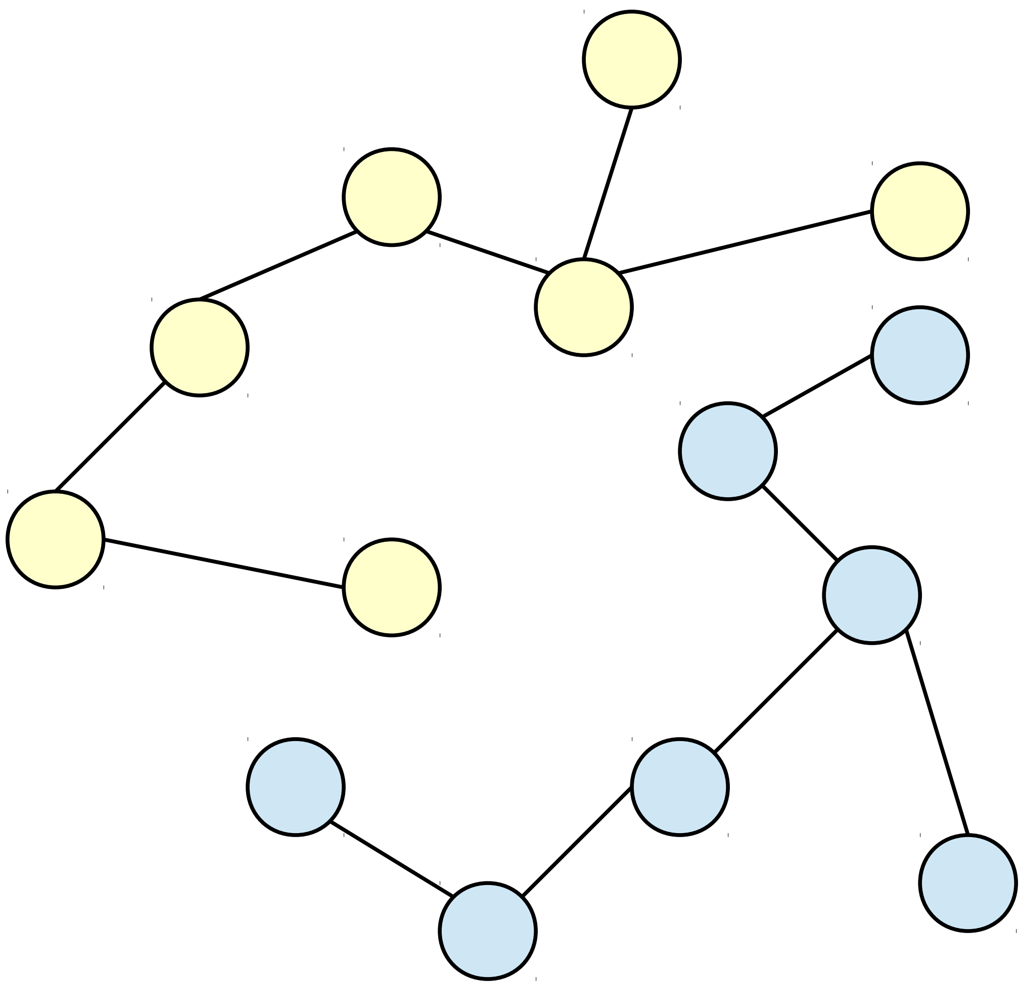

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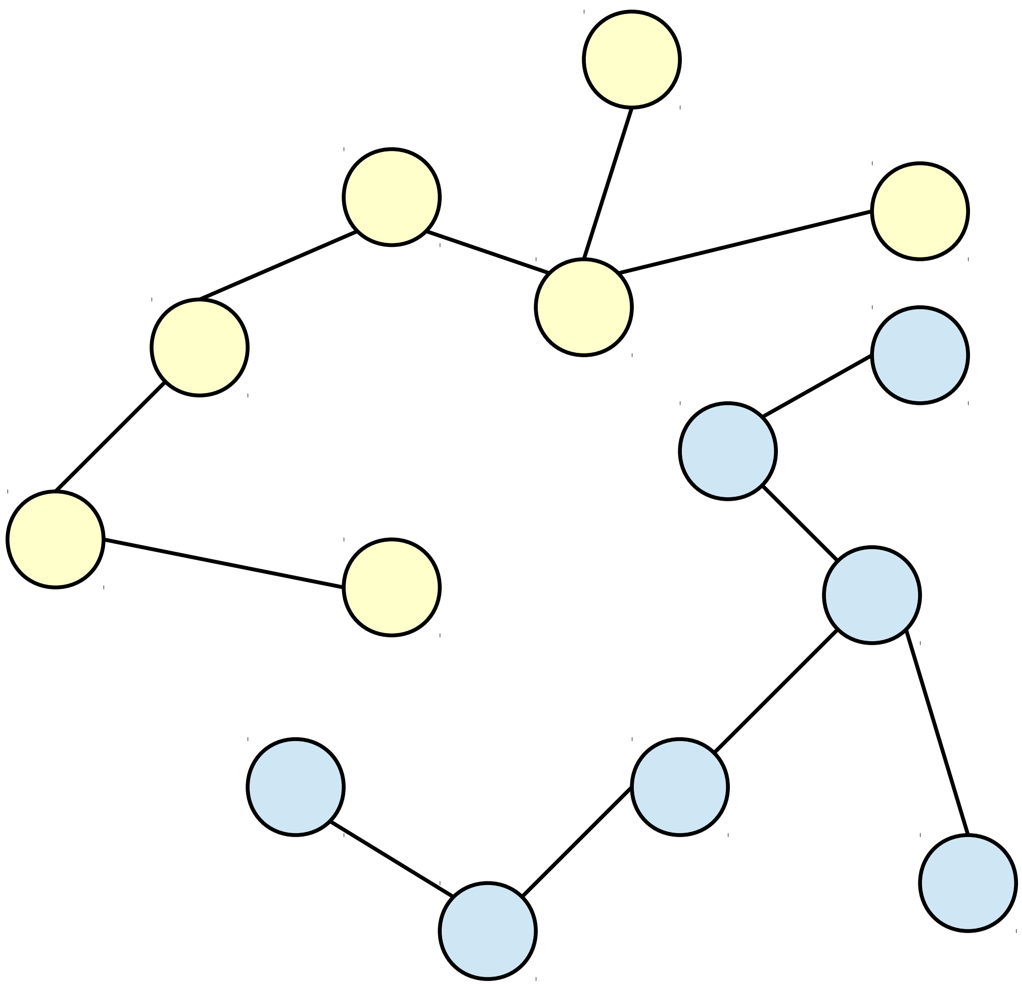

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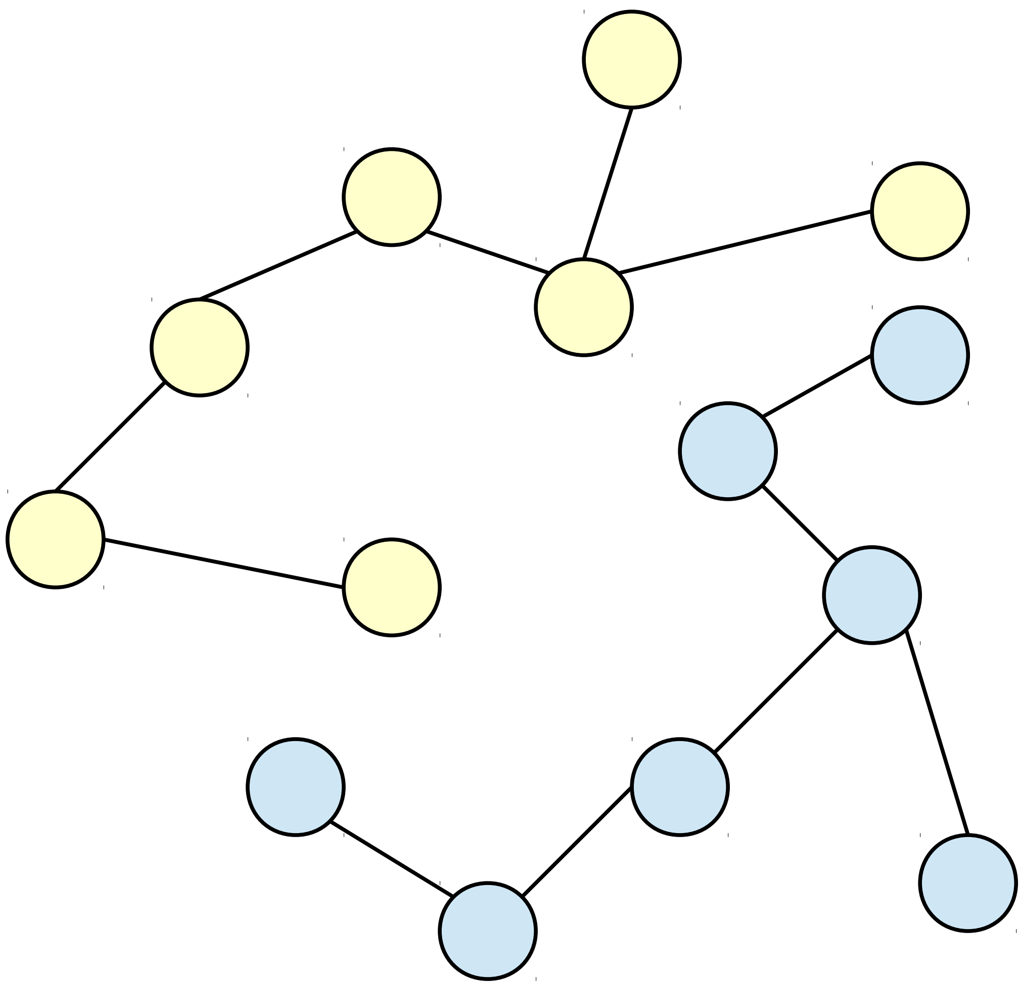
Then there are (*k*+1)-*r* nodes in the blue tree.



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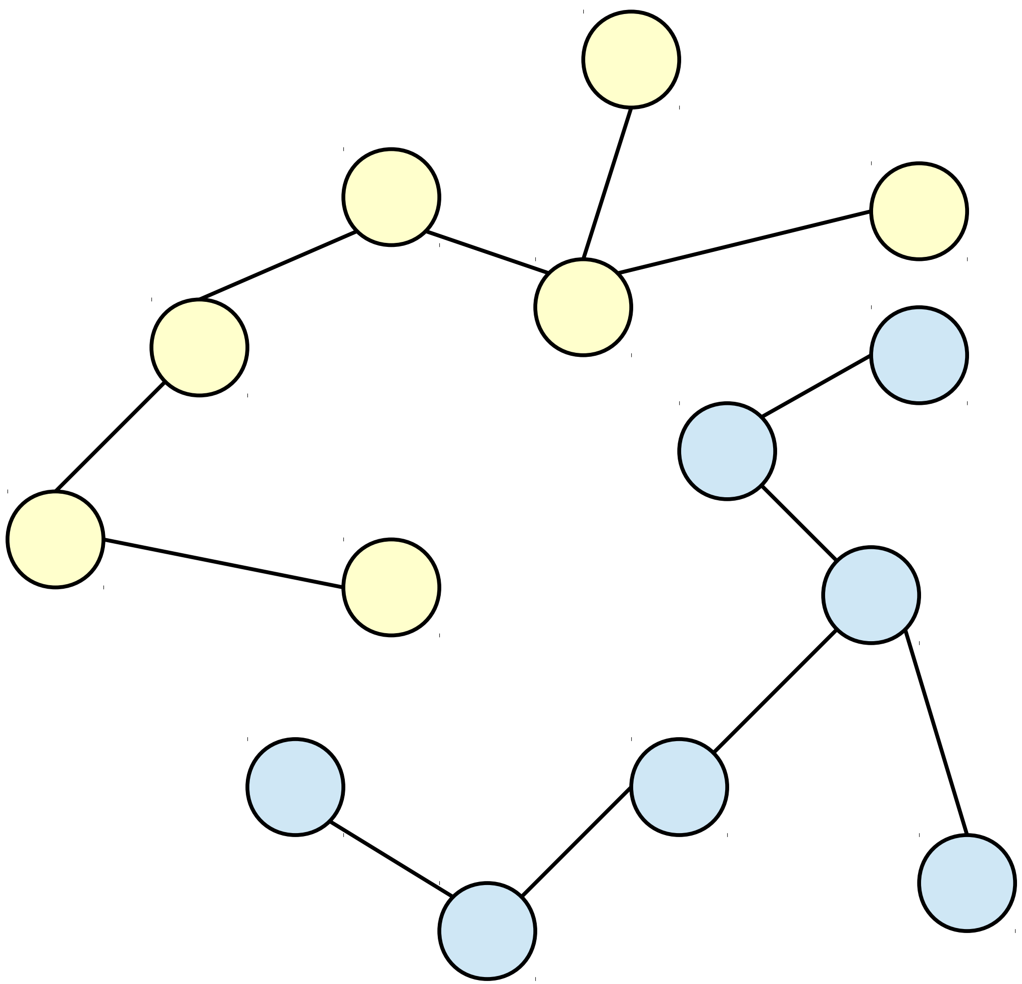

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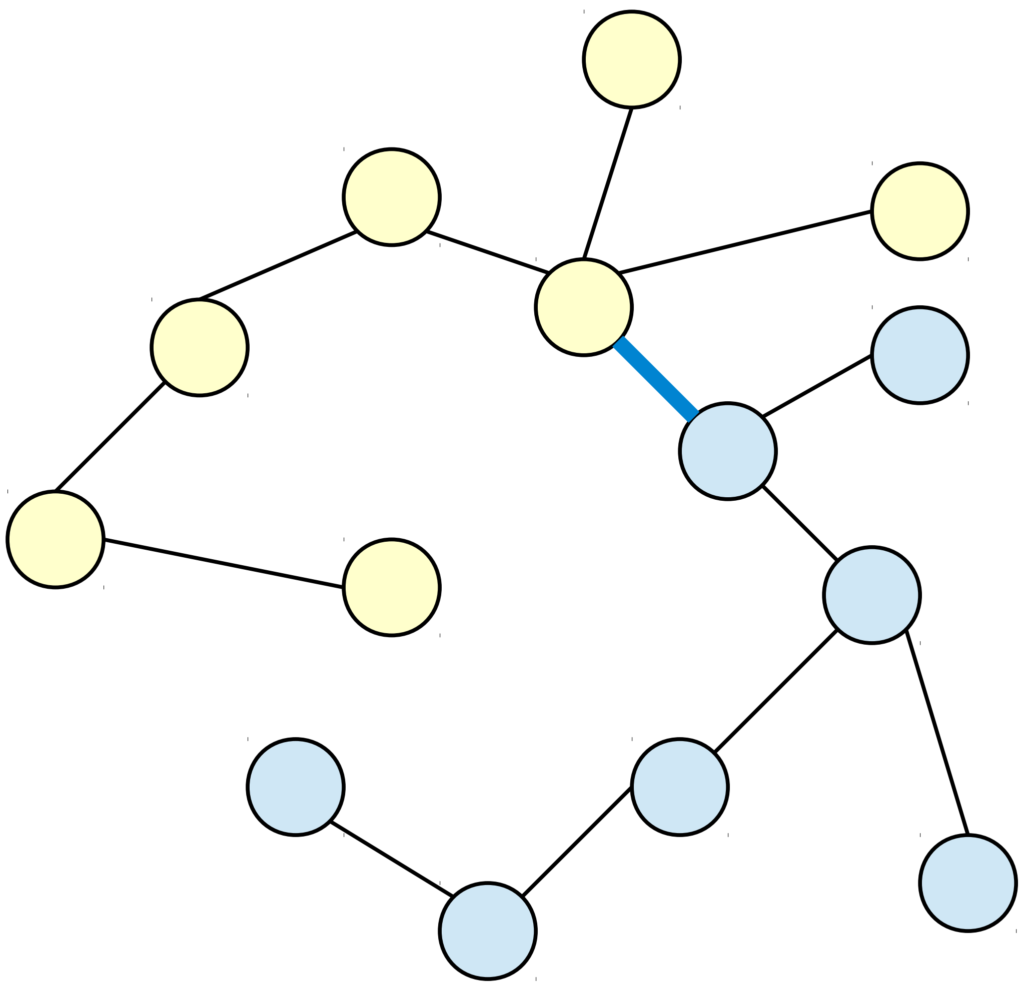

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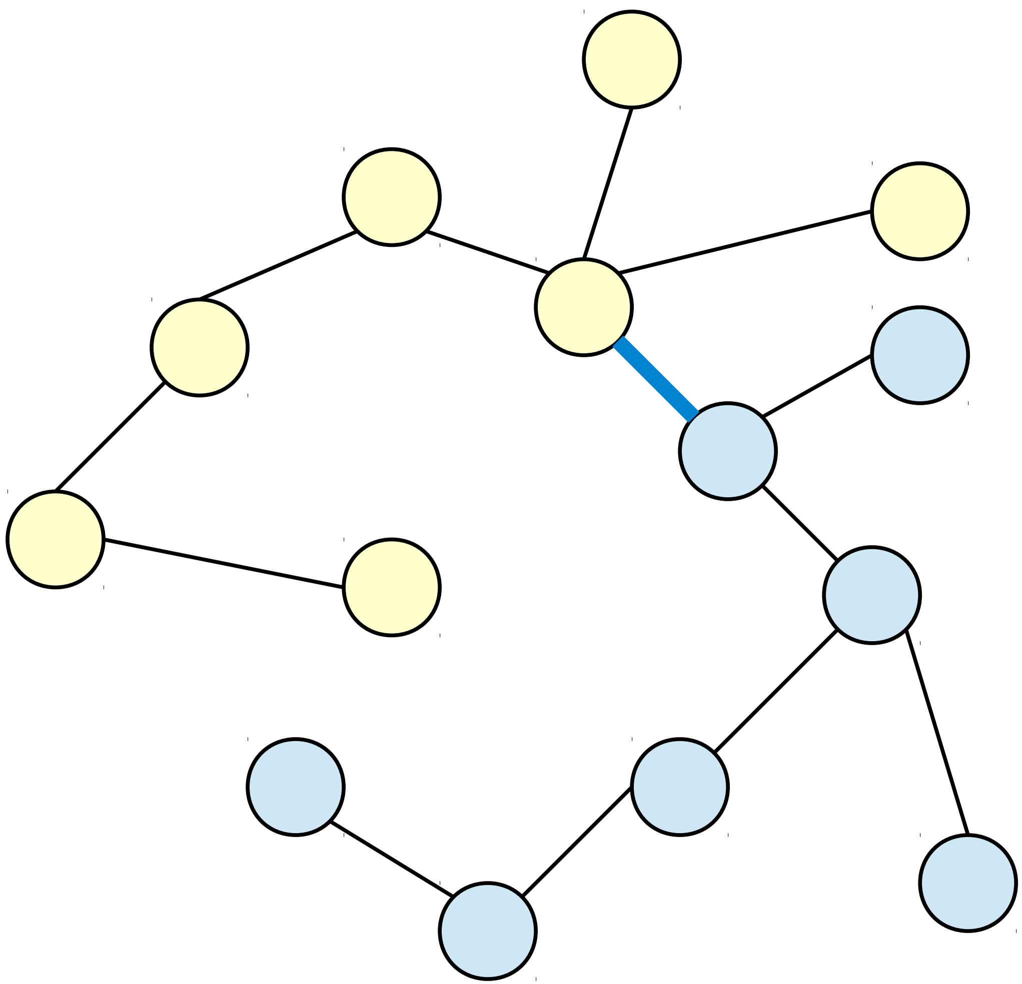
Adding in the initial edge we cut, there are r-1 + k-r + 1 = k edges in the original tree.



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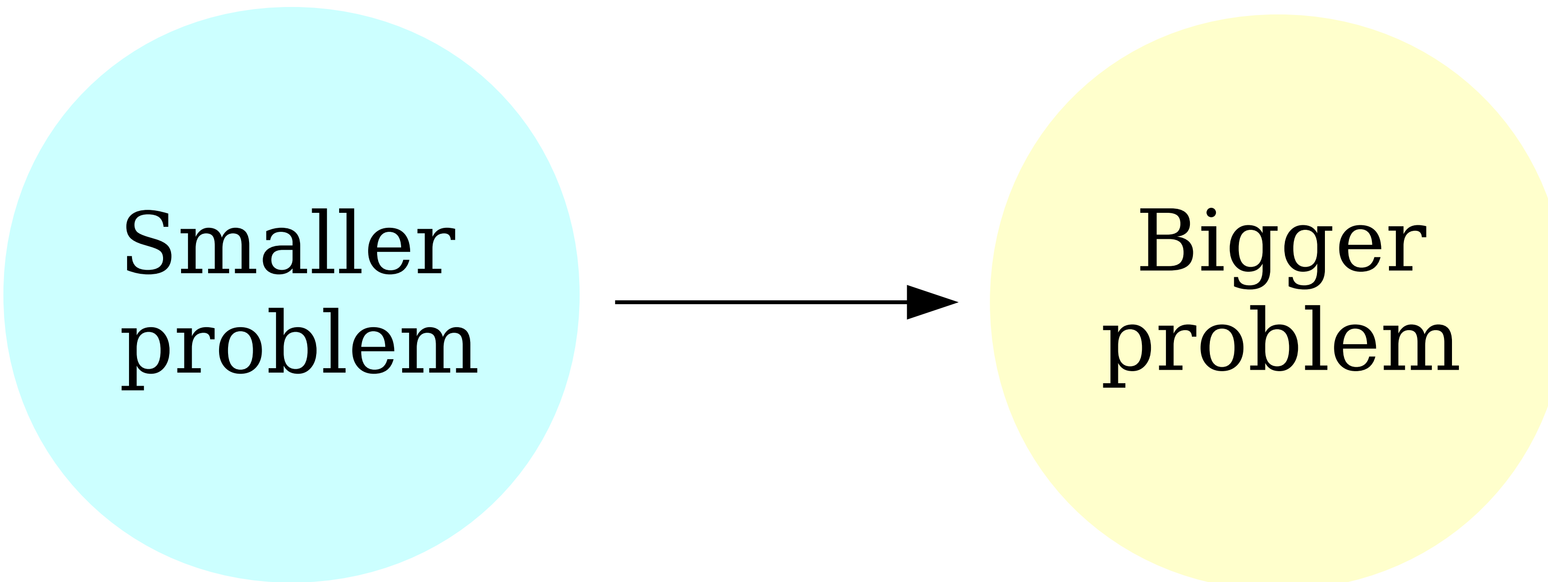
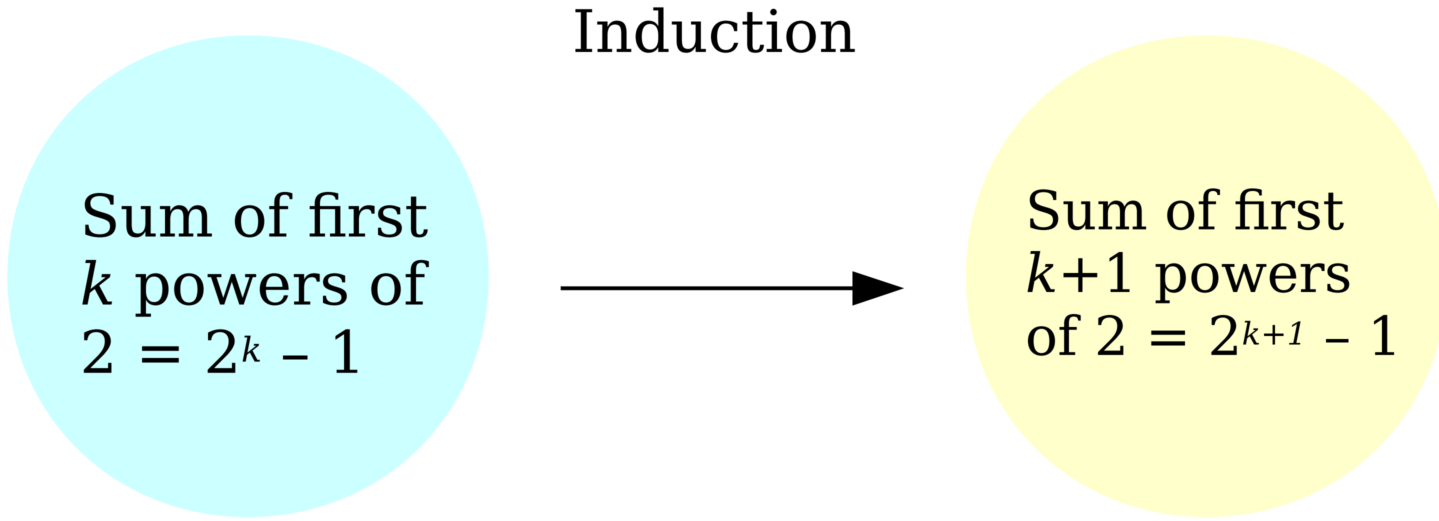
Induction vs. Complete Induction



Induction vs. Complete Induction

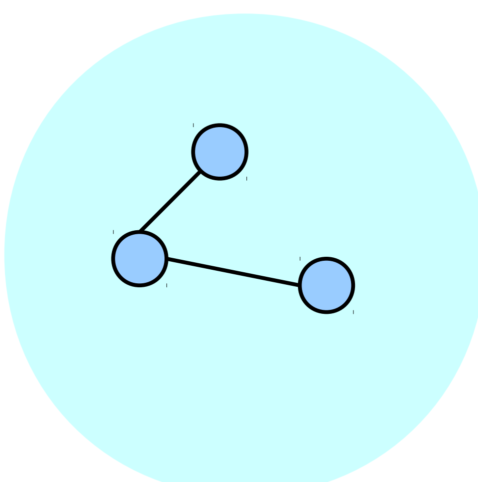


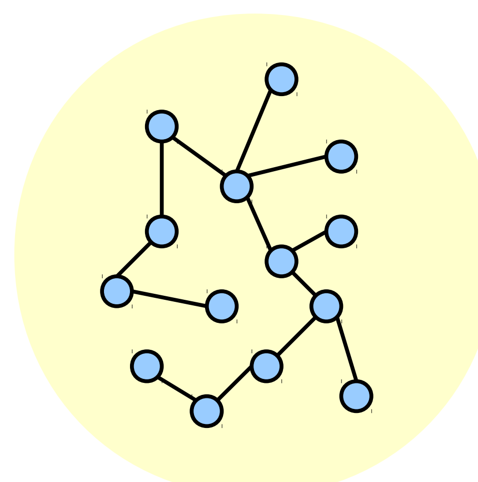
Induction vs. Complete Induction

Induction

Sum of first k powers of $2 = 2^k - 1$ → Sum of first $k+1$ powers of $2 = 2^{k+1} - 1$

Complete Induction





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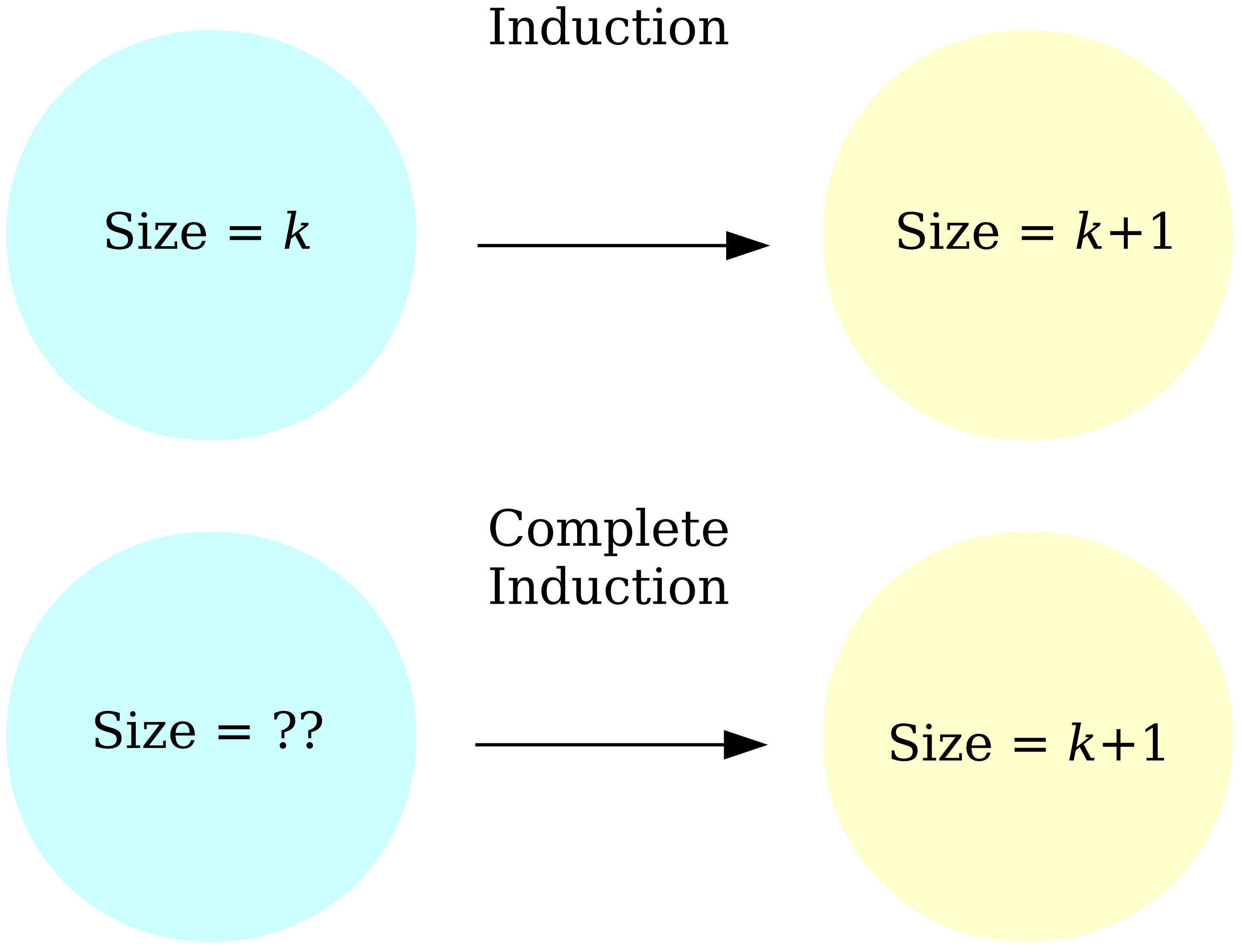
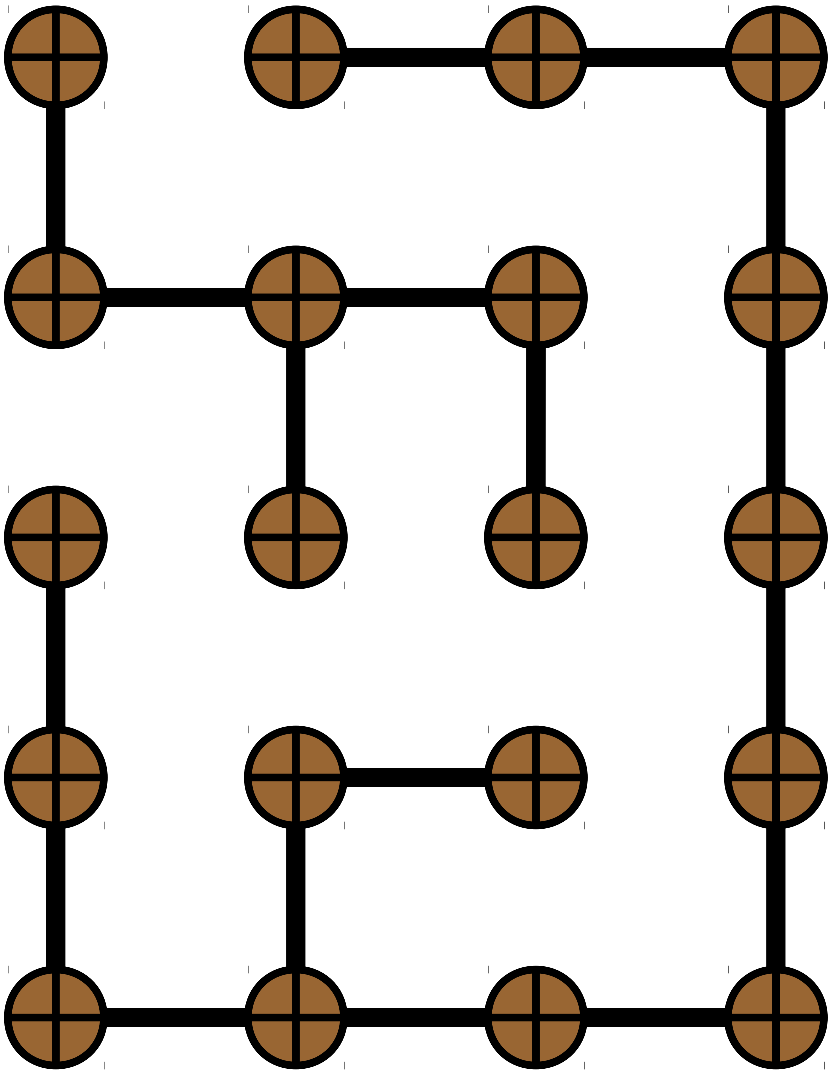


Rat Mazes

- Suppose you want to make a rat maze consisting of an $n \times m$ grid of pegs with slats between them.
- ***Question:*** How many slats do you need to create?



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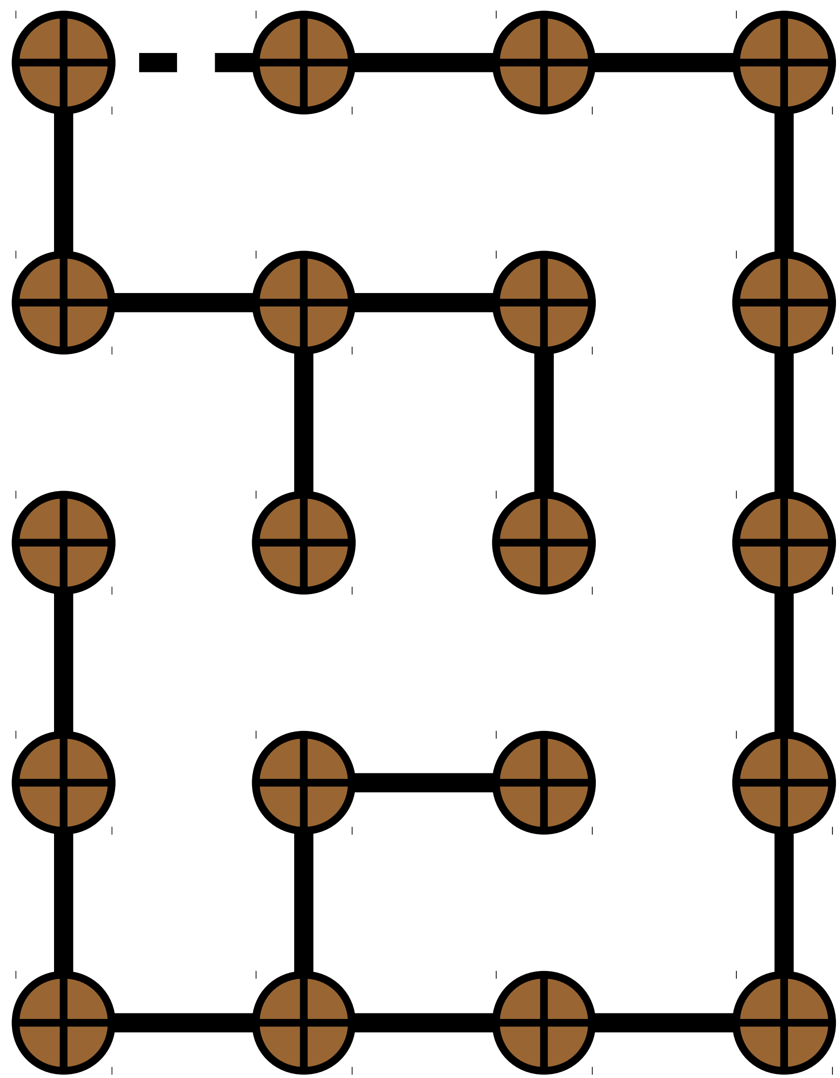

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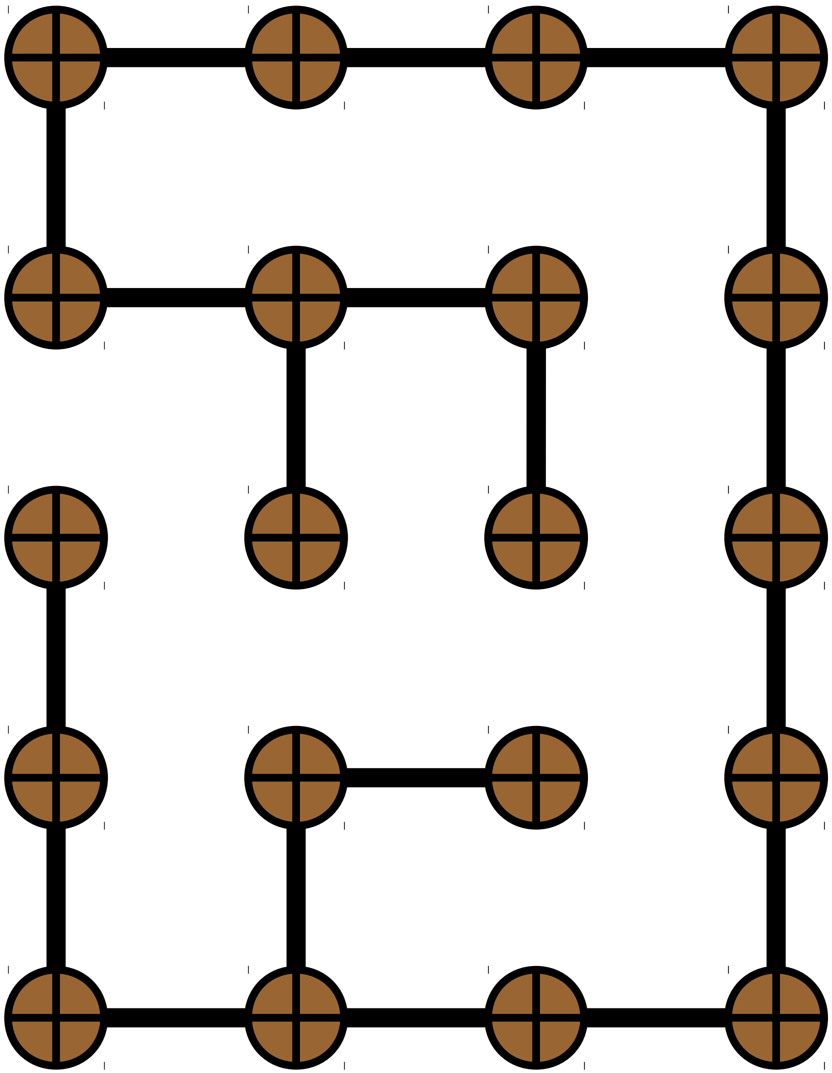

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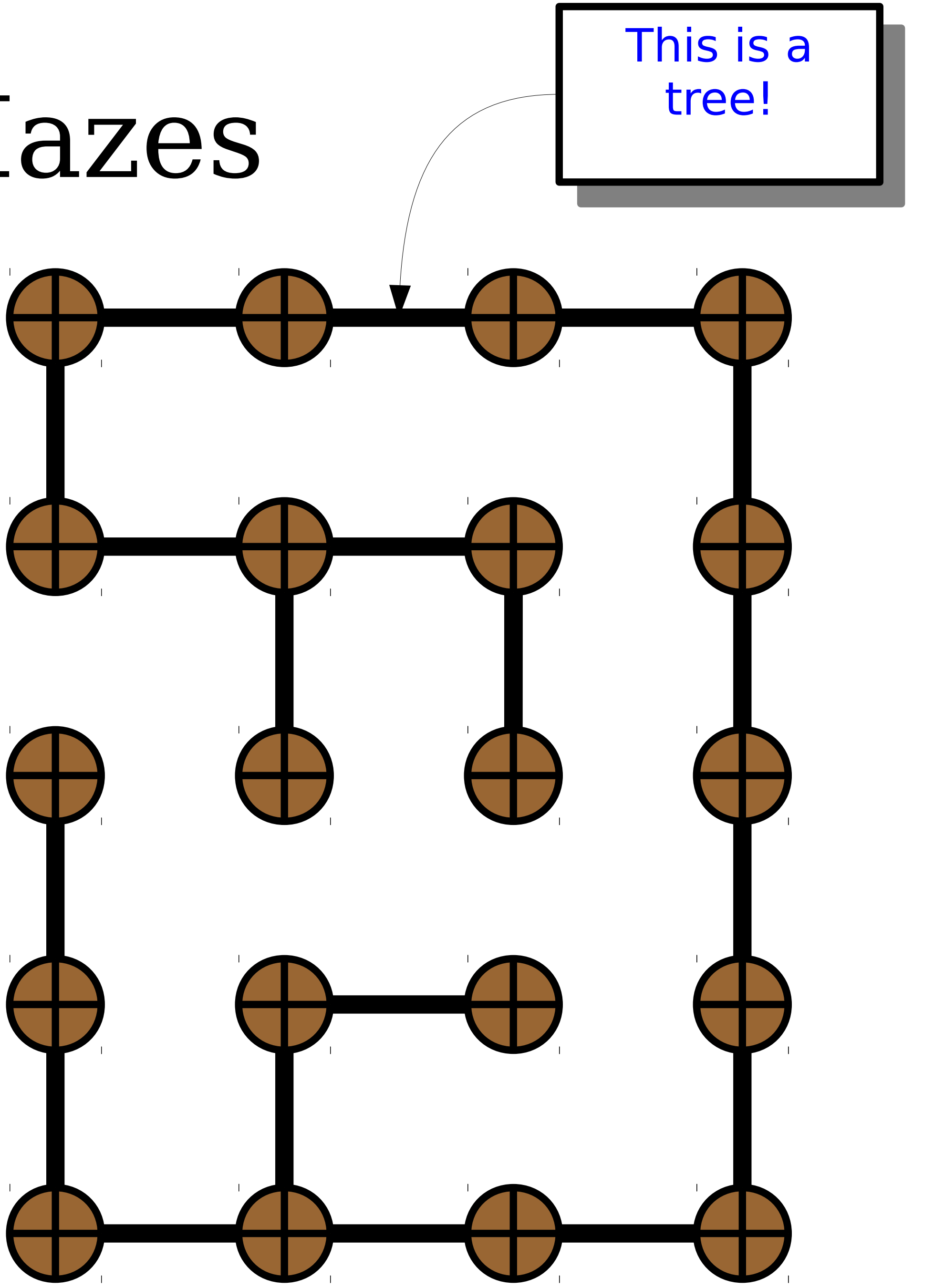
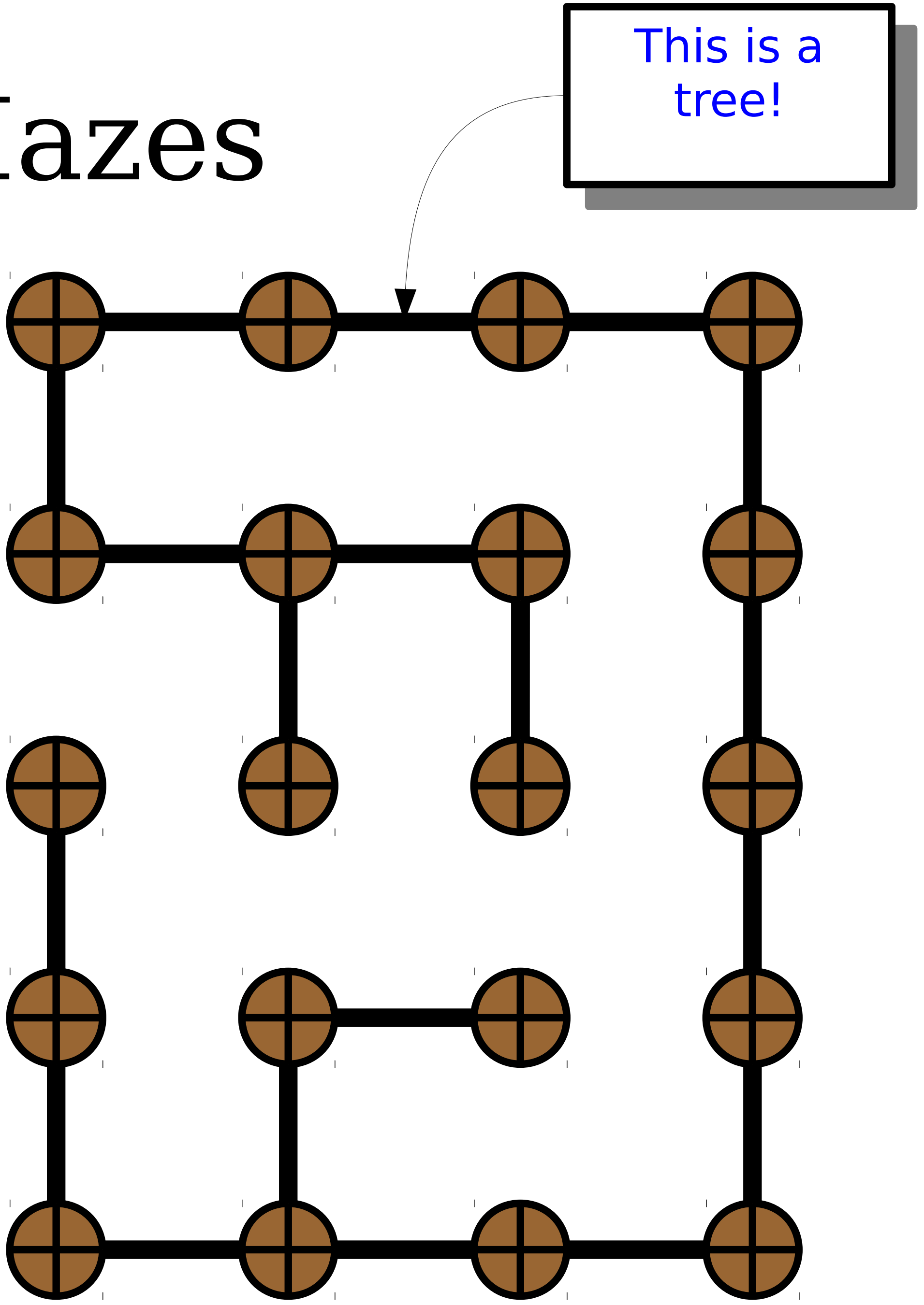


Rat Mazes

- Suppose you want to make a rat maze consisting of an $n \times m$ grid of pegs with slats between them.
- ***Question:*** How many slats do you need to create?
- ***Answer:*** $mn - 2$.



For more on trees, take CS161 / 261 / 267!

An Important Milestone

Recap: ***Discrete Mathematics***

- The past five weeks have focused exclusively on discrete mathematics:

Induction	Functions
Graphs	The Pigeonhole Principle
Relations	Mathematical Logic
Set Theory	Cardinality

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Next Up: ***Computability Theory***

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.